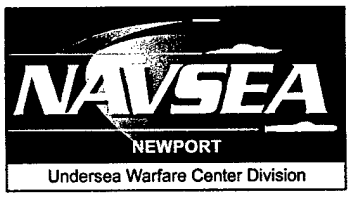
NUWC-NPT Technical Report 11,382
15 July 2002

Locating an Appropriate Saddlepoint for M-Dimensional Probability Integrals

Albert H. Nuttall
Surface Undersea Warfare Department



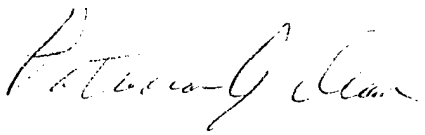
Naval Undersea Warfare Center Division
Newport, Rhode Island

PREFACE

The work described in this report was funded under Project No. A101402, "Automatic Signal Classification," principal investigator Stephen G. Greineder (Code 2121). The sponsoring activity is the Office of Naval Research, program manager John Tague (Code 321US).

The technical reviewer for this report was Paul M. Baggenstoss (Code 2121).

Reviewed and Approved: 15 July 2002



Patricia J. Dean
Head, Surface Undersea Warfare Department

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	15 July 2002	

4. TITLE AND SUBTITLE

Locating an Appropriate Saddlepoint for M-Dimensional Probability Integrals

5. FUNDING NUMBERS

6. AUTHOR(S)

Albert H. Nuttall

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Naval Undersea Warfare Center Division
1176 Howell Street
Newport, RI 02841-1708

8. PERFORMING ORGANIZATION REPORT NUMBER

TR 11,382

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Office of Naval Research
800 N. Quincy Street
Arlington, VA 22217-5660

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (Maximum 200 words)

The evaluation of the joint probability density function from the joint moment generating function involves an M-dimensional inverse Laplace transform. The analytic and numerical difficulty of performing this task for large values of M prompts consideration of an approximate technique such as the saddlepoint method. Advantage can be taken of the fact that the joint probability density function is real and positive, to show that the dominant saddlepoint in the original region of analyticity of the joint moment generating function is on the real axes. Furthermore, inside this region of analyticity, the integrand of the inverse Laplace transform has a positive-definite Hessian matrix on these real axes, indicating a single minimum for the saddlepoint location, when it exists. These properties serve to reduce the numerical effort required to locate the dominant M-dimensional saddlepoint. Examples of some statistical problems where this issue is of importance are included.

14. SUBJECT TERMS			15. NUMBER OF PAGES
Saddlepoint	Moment Generating Function	M-Dimensional	73
Region of Analyticity	Joint Probability	Positive-Definite Hessian	16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
Unclassified	Unclassified	Unclassified	SAR

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS

a	Average of g(z), equation (8)
a_{mn}	Transformation matrix elements, equation (19)
A	MxM transformation matrix, equation (105)
AC	Analytic continuation
$b_n(\lambda)$	n-th MGF function, equations (21), (86)
BC	Bromwich contour
BP	Branch point
c	Joint cumulative distribution function, equation (43)
c_{3a}, c_{3b}	Third-order parameters, equations (81), (82)
c_4	Fourth-order parameter, equation (80)
C	Contour of integration, equations (3), (37)
CF	Characteristic function
CGF	Cumulant generating function
C_λ	Contour to left of origin, equation (36)
C_r	Contour to right of origin, equation (54)
d_n	n-th constant, equation (106)
f	Characteristic function, equation (111)
g(u)	Nonlinear transformation, equations (8), (32)
$g_m(u)$	Component nonlinearities, equations (31), (38)

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

$h_n(x)$	n-th nonlinear transformation, equation (27)
HM	Hessian matrix
H	Hessian matrix of **x**, equations (67), (77)
Im	Imaginary part
m(n)	n-th integer, equation (106)
M	Number of variables or dimensions
MD	M-dimensional
MGF	Moment generating function
N	Number of independent random variables, equation (18)
p_m	m-th integer power, equation (73)
p(z)	Joint probability density function at z, equation (1)
$p_n(x)$	Approximations to p(x), equations (78), (79)
PD	Positive definite
PDF	Probability density function
Re	Real part
ROA	Complex region of analyticity
RV	Random variable or random vector
R_γ	Real region of existence of $\gamma(\lambda_r)$, equation (11)
R_μ	Real region of existence of MGF $\mu(\lambda_r)$, equation (1)
$R_{\mu\gamma}$	Real region of intersection, equation (14)
s	Change of variable, equation (70)
S_m	m-th set of integers, equation (108)
SP	Saddlepoint
SPA	Saddlepoint approximation
t	Change of variable, equation (64)

LIST OF ABBREVIATIONS, ACRONYMS, AND SYMBOLS (Cont'd)

T	Inverse matrix to H, equation (77)
u	Vector argument of transformation g(u), equation (8)
U	Unit step function, equation (9)
$v_j(m)$	Individual component of vector V(m), equation (112)
V	Matrix of vectors {V(m)}, equation (113)
$V(m)$	m-th component column vectors, equation (106)
w	N-dimensional random vector, equation (83)
x_n	n-th independent random variable, equation (18)
y	M-dimensional field point, equation (25)
y_m	m-th transformed random variable, equation (19)
z	M-dimensional field point, equation (1)
z_m	m-th component of $z = [z_1 \cdots z_M]^T$
z	M-dimensional random vector, equation (8)
z_m	m-th component of random vector z
$\Psi(\lambda)$	Integrand of general average, equation (15)
$\Psi(\lambda,z)$	Integrand of inverse Laplace transform, equation (4)
$\gamma(\lambda)$	Laplace transform of g(u), equation (11)
λ	M-dimensional vector in MGF space, equation (1)
λ_i	Imaginary component of vector λ, equation (2)
λ_m	m-th component of $\lambda = [\lambda_1 \cdots \lambda_M]^T$
λ_r	Real component of vector λ, equation (2)
$\hat{\lambda}, \hat{\lambda}(z)$	Location of M-dimensional saddlepoint, equation (5)
$\Lambda(\lambda)$	Logarithm of integrand $\Psi(\lambda)$, equation (16)
$\Lambda(\lambda,z)$	Logarithm of integrand $\Psi(\lambda,z)$, equation (7)
$\mu(\lambda)$	Joint moment generating function, equation (1)

$\chi(\lambda)$	Joint cumulant generating function, equation (7)
ν_m	m-th parameter of g(u), equation (10)
Superscript T	Transpose of vector
overbar	Ensemble average, equation (18)
boldface	Random variable or random vector

LOCATING AN APPROPRIATE SADDLEPOINT FOR M-DIMENSIONAL PROBABILITY INTEGRALS

INTRODUCTION

The M-dimensional (MD) moment generating function (MGF) for M dependent random variables (RVs) of interest can sometimes be found in closed form; for example, see references 1 and 2. However, this joint MGF is usually not the desired end product of an analysis. Instead, its MD inverse Laplace transform, namely, the joint probability density function (PDF), might be the quantity of interest. Alternatively, the joint MGF might be weighted by an MD function prior to performing the MD inverse Laplace transform, thereby yielding the joint cumulative distribution function, or the joint exceedance distribution function, or some other statistics of the RVs of interest; see reference 2, equation (18) and examples 1 through 7 for several informative illustrations of this general approach.

Since the joint MGF frequently exists only in strips of analyticity in its MD domain, the MD Bromwich contour (BC) for returning to the probability domain must initially be taken in this region of existence of the joint MGF. However, at least for one-dimensional problems, M = 1, the MGF can be analytically continued outside this strip, and the original BC can be moved to a more attractive region of the complex plane, taking advantage

of possible saddlepoints (SPs) and/or paths of steepest descent of the analytic continuation of the MGF. In fact, for many one-dimensional examples, an exact result for the first-order PDF can be obtained by moving the BC to an appropriate valley at infinity, while accounting for any residues of poles or essential singularities traversed during the contour movement.

For M dimensions, the situation is much more complicated. The analytic difficulties associated with multiple complex domains and the movement of multiple contours discourages and often prevents any useful improvement in trying to move the BCs out of their original MD domain of definition. Instead, a more fruitful approach is to try to find a single dominant SP of the pertinent integrand in the original MD strip of analyticity and develop a saddlepoint approximation (SPA) (plus a correction term if possible) about this point in MD space. However, it should be noted that there is a question about the existence of an SP in the original strip of analyticity; this problem will be illustrated in this report later.

Even if a single dominant SP exists in the MD strip of analyticity, there remains the numerical problem of specifying and quantifying its location in complex MD space. It will be shown that this MD search procedure can be restricted to MD real space on a real scalar function that has a monotonic, bowl-like behavior, thereby guaranteeing a single minimum in MD space.

LOCATING AN APPROPRIATE SADDLEPOINT IN MOMENT GENERATING SPACE FOR EVALUATION OF A PROBABILITY DENSITY FUNCTION

Let MD real vector $z = [z_1 \cdots z_M]^T$. Scalar function $p(z)$ is an MD joint PDF; that is, $p(z)$ is real, nonnegative, and has unit volume in MD z space. Let MD complex vector $\lambda = \lambda_r + i\ \lambda_i$ in terms of its real and imaginary MD component vectors, λ_r, λ_i. The MD Laplace transform of $p(z)$ is denoted as the scalar joint MGF

$$\mu(\lambda) = \int dz\ p(z)\ \exp(\lambda^T z)\ . \tag{1}$$

For λ purely real, that is, $\lambda_i = 0$, it is presumed that integral (1) converges for real vector λ_r in an MD real region R_μ that includes the origin, $\lambda_r = 0$, as an interior point in MD λ_r space. The joint MGF $\mu(\lambda_r)$ is obviously positive real for all $\lambda_r \in R_\mu$.

It follows that if $\mu(\lambda_r)$ exists, then MGF $\mu(\lambda) = \mu(\lambda_r + i\ \lambda_i)$ also exists for all λ_i, as seen from the relation

$$\left|\mu(\lambda_r + i\ \lambda_i)\right| = \left|\int dz\ p(z)\ \exp\left(\lambda_r^T z + i\ \lambda_i^T z\right)\right|$$

$$\leq \int dz\ p(z)\ \exp\left(\lambda_r^T z\right) = \mu(\lambda_r) \quad \text{for all } \lambda_i\ . \tag{2}$$

This MD region of complex λ is the region of analyticity of joint MGF $\mu(\lambda)$ and is denoted by ROA(μ); that is, ROA(μ) is the MD set of complex λ values such that $\mathrm{Re}(\lambda) \in R_\mu$. Joint MGF $\mu(\lambda)$ does not exist outside of ROA(μ), although its analytic continuation (AC) may. A one-dimensional example is furnished by PDF $p(z) = \frac{1}{2} \exp(-|z|)$ for all z; the corresponding MGF is $\mu(\lambda) = 1/(1-\lambda^2)$ for $-1 < \lambda_r < 1$. The AC of $\mu(\lambda)$ is $\mu_{ac}(\lambda) = 1/(1-\lambda^2)$ for all

$\lambda \neq \pm 1$, at which points $\mu_{ac}(\lambda)$ has poles.

Equation (2) also indicates that, for a fixed real vector $\lambda_r \in R_\mu$, the maximum value of $|\mu(\lambda)|$ anywhere on the M BCs, $\lambda = \lambda_r + i\,\lambda_i$, is realized as all the contours simultaneously cross the M real axes in λ space. An alternative interpretation is that the real axes in λ space are mountain ridges of joint MGF $|\mu(\lambda)|$; any perpendicular departure from the real axes in any or all of the M complex planes $\{\lambda_m\}$ of vector λ, leads to a decrease in $|\mu(\lambda)|$ when $\lambda \in \text{ROA}(\mu)$.

The joint PDF p(z) at real MD field point z can be recovered from the joint MGF $\mu(\lambda)$ according to inverse Laplace transform

$$p(z) = \frac{1}{(i2\pi)^M} \int_C d\lambda\ \mu(\lambda)\ \exp(-\lambda^T z)\ , \tag{3}$$

where MD contour C is a set of vertical BCs that stay in the ROA(μ). The integrand in equation (3) is denoted as

$$\Psi(\lambda,z) = \mu(\lambda)\ \exp(-\lambda^T z)\ . \tag{4}$$

The exponential can be expressed as $\exp(-\lambda_r^T z)\ \exp(-i\ \lambda_i^T z)$, which retains a constant magnitude as vector λ_i alone is varied. Therefore, integrand $\Psi(\lambda,z)$ in equations (4) and (3) realizes its maximum magnitude on the BCs at the real points where the individual contours all cross the real axes in λ space. Therefore, the major contribution to integral (3) is expected to occur in the neighborhood of the points where the M BCs cross the

real axes in λ space.

Since Cauchy's integral theorem states that the value of integral (3) is the same for any set of vertical BCs in the ROA(μ), it is possible to choose, for the crossing point, that MD real point in λ_r space where the positive real integrand $\Psi(\lambda_r,z)$ is a global minimum; this MD real point is identified as $\hat{\lambda} = \hat{\lambda}(z)$. All other axis crossings encounter a larger peak magnitude of integrand $\Psi(\lambda,z)$, and therefore must also encounter more oscillations in the integrand away from the crossing point in order to compensate and maintain the identical integral value for PDF (3) at that z value. Selection of the crossing point $\hat{\lambda}(z)$ as the minimum of real integrand $\Psi(\lambda_r,z)$ essentially minimizes the oscillations of complex integrand $\Psi(\lambda,z)$ on the vertical BCs in the neighborhood of the crossing point and partially realizes the steepest decay of the magnitude of the integrand $\Psi(\hat{\lambda} + i\,\lambda_i,z)$. However, the selection of the location of the minimum of $\Psi(\lambda_r,z)$ as the crossing point $\hat{\lambda}$ to be used for the BCs is not mandatory; it is taken mainly for convenience and simplicity. An alternative for a one-dimensional example involving a branch point (BP) at the border of R_μ is given in appendix A.

This particular real axes crossing point, $\hat{\lambda}(z)$, is the location of a maximum of $|\Psi(\lambda,z)|$ along every BC, and is the location of a simultaneous minimum of $\Psi(\lambda_r,z)$ along every real axis in λ space. If all the slopes of $\Psi(\lambda_r,z)$ are zero at

$\hat{\lambda} = \hat{\lambda}(z)$, this crossing point is an SP in MD λ space. Thus,

$$\frac{\partial}{\partial \lambda_m} \Psi(\lambda, z) \Big|_{\hat{\lambda}} = 0 \quad \text{for } m=1:M \ , \quad \hat{\lambda} \text{ real} \ , \tag{5}$$

specifies the location of a real SP in R_μ. This set of M simultaneous, nonlinear real equations must be solved numerically for real $\hat{\lambda} \in R_\mu$; if a solution exists, $\hat{\lambda} = \hat{\lambda}(z)$ is a function of the field point z of interest.

If equations (3) and (4) are combined, there follows

$$p(z) = \frac{1}{(i2\pi)^M} \int_C d\lambda \, \exp[\Lambda(\lambda, z)] \ , \tag{6}$$

where

$$\Lambda(\lambda, z) = \ln \Psi(\lambda, z) = \ln \mu(\lambda) - \lambda^T z \equiv \chi(\lambda) - \lambda^T z \ , \tag{7}$$

and where $\chi(\lambda)$ is the joint cumulant generating function (CGF) corresponding to joint MGF $\mu(\lambda)$. The Hessian matrix (HM) of joint CGF $\chi(\lambda)$ is the MxM matrix of second-order partial derivatives of $\chi(\lambda)$ with respect to $\{\lambda_m\}$, m=1:M, which are the M complex components of MD vector λ. It can be seen from equation (7) that the HM of $\Lambda(\lambda, z)$ is identical to that of $\chi(\lambda)$, and is independent of field point z. It is shown in appendix B that the HM of joint CGF $\chi(\lambda)$ is positive definite (PD) for λ real and in R_μ. Therefore, the HM of $\Lambda(\lambda, z)$ in equation (7) is also PD for λ real and in R_μ. This means that $\Lambda(\lambda_r, z)$ has positive curvature for all $\lambda_r \in R_\mu$; that is, $\Lambda(\lambda_r, z)$ has a bowl-like behavior and

can have, at most, one minimum for $\lambda_r \varepsilon R_\mu$. More details are presented in appendix C.

From equation (7), since the integrand of equation (6) is related to $\Lambda(\lambda_r,z)$ by the monotonically increasing transformation $\Psi(\lambda_r,z) = \exp[\Lambda(\lambda_r,z)]$, the same bowl-like property carries over to the integrand. That is, real positive integrand $\Psi(\lambda_r,z)$ can have, at most, one minimum for $\lambda_r \varepsilon R_\mu$. The location of this minimum (if it has zero slopes as required by equation (5)) is the unique <u>real</u> SP of $\Psi(\lambda,z)$ in the ROA(μ). This real SP $\hat{\lambda}(z)$ can be used to connect the valleys of the integrand $\Psi(\lambda,z)$ at $\lambda = \pm i\infty$ by means of BCs. Then, an SPA to p(z) at field point z can be developed about this real SP $\hat{\lambda}(z)$ in MGF space.

There may be other complex SPs of integrand $\Psi(\lambda,z)$ for λ in the ROA(μ); a one-dimensional example is furnished by PDF $p(z) = A \exp(-z)\,[1 + \cos(az)]$ for $z > 0$. For $a = 1$ and $z = 0.1$, the integrand $\Psi(\lambda,z)$ has a real SP at $\hat{\lambda}(z) = -8.8995$ and a pair of complex SPs at $0.4951 \pm i\ 0.70305$, all of which are in the ROA(μ); namely, $\lambda_r < 1$. The AC of $\Psi(\lambda,z)$ has another pair of complex SPs at $\lambda(z) = 1.4547 \pm i\ 0.68369$ that are outside the ROA(μ). More details on this example are furnished in appendix D.

For some values of z, a real SP of $\Psi(\lambda,z)$ may not exist in the ROA(μ). For example, for one-dimensional PDF $p(z) = \exp(-z)$ for $z > 0$, then MGF $\mu(\lambda) = 1/(1 - \lambda)$ for $\lambda_r < 1$. Then, if field point z is taken negative, $z < 0$, the real integrand $\Psi(\lambda_r,z)$ is a

strictly monotonically increasing function of λ_r for all λ_r in $(-\infty,1)$ and never has zero slope there. This particular value of z is unfeasible; that is, z corresponds to a zero value for the PDF p(z). The case of z = 0+ has an SP at $\lambda_r = -\infty$. Figure 1 illustrates this behavior for function $\Lambda(\lambda_r,z)$.

But, even if field point z is a feasible point, the integrand $\Psi(\lambda,z)$ may still not have a real SP in R_μ, even though $\Psi(\lambda_r,z)$ has a unique minimum for $\lambda_r \in R_\mu$. A one-dimensional example of this situation is furnished by the PDF $p(z) = A \exp(-z)/(1+z)^m$ for $z > 0$. More details on this example are given in appendix E. Thus, each candidate real solution of equation (5) must be tracked as the search for the unique real SP in R_μ develops in order to ensure that the search is not tending to infinity or attempting to get out of region R_μ.

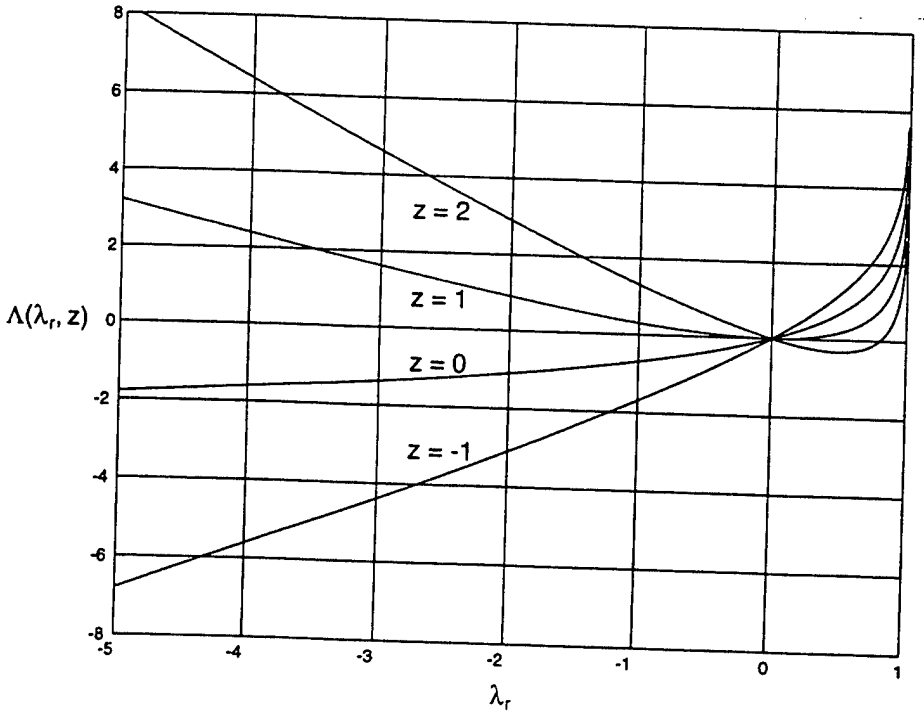


Figure 1. Logarithm of Integrand for Exponential PDF

LOCATING AN APPROPRIATE SADDLEPOINT IN MOMENT GENERATING SPACE FOR EVALUATION OF A NONLINEARITY AVERAGE

The presentation in this section is somewhat similar to that of the previous section and will be abbreviated. Suppose random MD vector z is subject to an MD nonlinear transformation $g(u) = g(u_1,...,u_M)$, giving scalar output g(z). The average of this random quantity is given by

$$a = E\{g(\mathbf{z})\} = \int du\ p(u)\ g(u)\ , \tag{8}$$

where p(u) is again the joint MD PDF of RV z. Transformation g(u) can be a function with additional parameters, such as the MD field point $z = [z_1\ \cdots\ z_M]^T$ of interest, as in

$$g(u) = \prod_{m=1}^{M} U(u_m - z_m)\ , \tag{9}$$

where U( ) is the unit-step function, or the power-law values $\{\nu_m\}$ in a transformation such as

$$g(u) = \prod_{m=1}^{M} \left(u_m^{\ \nu_m}\right) \quad \text{for } u_m > 0\ ,\ m=1:M\ . \tag{10}$$

Then, the average a in equation (8) will be a function of these additional parameters. Function g(u) is presumed to be real and nonnegative for all u; however, g(u) need not have finite volume in u space, as illustrated by the two examples above. Additional interpretations and examples are given in reference 2.

The MD Laplace transform of nonlinearity g(u) is denoted by

$$\gamma(\lambda) = \int du\ g(u)\ \exp(-\lambda^T u)\ , \tag{11}$$

with a minus sign in the exp( ); this minus is in contrast to the Laplace transform for the joint MGF $\mu(\lambda)$ in equation (1), where a plus sign was used. For λ purely real, that is, $\lambda_i = 0$, it is presumed that integral (11) converges for λ_r in an MD region R_γ. The function $\gamma(\lambda_r)$ is obviously positive real for all $\lambda_r \in R_\gamma$.

It follows that if $\gamma(\lambda_r)$ exists, then $\gamma(\lambda) = \gamma(\lambda_r + i\ \lambda_i)$ also exists for all λ_i, as seen from the relation

$$\left|\gamma(\lambda_r + i\ \lambda_i)\right| = \left|\int du\ g(u)\ \exp\left(-\lambda_r^T u - i\ \lambda_i^T u\right)\right|$$

$$\leq \int du\ g(u)\ \exp\left(-\lambda_r^T u\right) = \gamma(\lambda_r) \quad \text{for all } \lambda_i\ . \tag{12}$$

This MD region of complex λ is the region of analyticity of $\gamma(\lambda)$, which is denoted by ROA(γ); that is, ROA(γ) is the MD set of complex λ values such that Re(λ) $\in R_\gamma$. Function $\gamma(\lambda)$ does not exist outside of ROA(γ), although its AC may. For example, function (9) yields

$$\gamma(\lambda) = \exp(-\lambda^T z) \Big/ \prod_{m=1}^{M} (\lambda_m) \quad \text{for } \mathrm{Re}(\lambda_m) > 0\ ,\ m=1:M\ . \tag{13}$$

That is, ROA(γ) here consists of the right halves of the M complex planes $\{\lambda_m\}$, m=1:M.

Equation (12) also indicates that, for a fixed real vector $\lambda_r \in R_\gamma$, the maximum value of $|\gamma(\lambda)|$ anywhere on the M BCs, $\lambda = \lambda_r + i\,\lambda_i$, is realized as all the contours simultaneously cross the M real axes in λ space. Any perpendicular departure from the real axes, in any or all of the M complex planes $\{\lambda_m\}$, leads to a decrease in $|\gamma(\lambda)|$ when $\lambda \in \mathrm{ROA}(\gamma)$.

Substitution of relation (3) into equation (8) and an interchange of integrals leads to an alternative result for average a; namely,

$$a = \frac{1}{(i2\pi)^M} \int_C d\lambda\ \mu(\lambda)\ \gamma(\lambda) \tag{14}$$

upon use of equation (11) (see reference 2 for several practical applications). However, this MD contour C must now stay in the intersection of ROAs; that is, $\lambda \in \mathrm{ROA}(\mu,\gamma) \equiv \mathrm{ROA}(\mu) \cap \mathrm{ROA}(\gamma)$ is required in equation (14). Also, define the MD real region $R_{\mu\gamma} = R_\mu \cap R_\gamma$.

The integrand in equation (14) is denoted as

$$\Psi(\lambda) = \mu(\lambda)\ \gamma(\lambda) \tag{15}$$

and will be a function of any additional parameters that nonlinearity g(u) depends on; for example, see equations (9) and (10). If MD contour C in equation (14) is taken as a set of BCs, integrand $\Psi(\lambda)$ will realize its maximum magnitude on C at

the real points where the individual contours all cross the real axes in λ space. Therefore, the major contribution to integral (14) is expected to occur in the neighborhood of the points where the M contours cross the real axes in λ space. This is the MD point about which to develop an SPA for average a of equations (8) and (14).

From equations (15) and (14), the logarithm of the integrand is given by

$$\Lambda(\lambda) \equiv \ln \Psi(\lambda) = \chi(\lambda) + \ln \gamma(\lambda) \ . \tag{16}$$

For $\lambda_i = 0$, it has been shown in appendix B that the HM of joint CGF $\chi(\lambda)$ is PD for $\lambda_r \in R_\mu \in R_{\mu\gamma}$. Similarly, appendix F demonstrates that the HM of $\ln \gamma(\lambda)$ is nonnegative definite for $\lambda_r \in R_\gamma \in R_{\mu\gamma}$. Therefore, the HM of sum $\Lambda(\lambda)$ in equation (16) must be PD for $\lambda_r \in R_{\mu\gamma}$. This observation follows from relation

$$v^T (A + B) v = v^T A v + v^T B v \ , \tag{17}$$

where v is an arbitrary real column vector, and A and B are real square matrices. Notice that complex vector λ must satisfy the restriction $\mathrm{Re}(\lambda) \in R_{\mu\gamma}$, while $\mathrm{Im}(\lambda)$ must be zero for the PD property of the HM of $\Lambda(\lambda)$ to hold.

In the following sections, some practical problems where these issues have relevance will be indicated.

JOINT MOMENT GENERATING FUNCTION OF M WEIGHTED SUMS OF N INDEPENDENT RANDOM VARIABLES

Let $\{x_n\}$ be a set of N independent real RVs with arbitrary first-order (different) MGFs $\{\mu_{x_n}\}$ and corresponding first-order CGFs $\{\chi_{x_n}\}$. That is,

$$\mu_{x_n}(\lambda) = \overline{\exp(\lambda\, x_n)}\ , \quad \chi_{x_n}(\lambda) = \ln \mu_{x_n}(\lambda) \quad \text{for } n=1:N\ , \tag{18}$$

where the overbar denotes an ensemble average. The weighted sums of independent RVs of interest are given by

$$y_m = \sum_{n=1}^{N} a_{mn}\, x_n \quad \text{for } m=1:M\ , \tag{19}$$

where $M \le N$, and $M \times N$ real matrix $[a_{mn}]$ is arbitrary except that it must have rank M.

The joint MGF of RVs $\{y_m\}$ is, upon use of equations (18) and (19) and the independence of RVs $\{x_n\}$,

$$\mu_y(\lambda_1,\ldots,\lambda_M) \equiv \overline{\exp\left(\sum_{m=1}^{M} \lambda_m\, y_m\right)} = \overline{\exp\left(\sum_{m=1}^{M} \lambda_m \sum_{n=1}^{N} a_{mn}\, x_n\right)}$$

$$= \prod_{n=1}^{N} \overline{\exp\left(x_n \sum_{m=1}^{M} a_{mn}\, \lambda_m\right)} = \prod_{n=1}^{N} \mu_{x_n}\left(b_n(\lambda)\right)\ , \tag{20}$$

where $M \times 1$ vector $\lambda = [\lambda_1 \cdots \lambda_M]'$ and

$$b_n(\lambda) \equiv \sum_{m=1}^{M} a_{mn}\, \lambda_m \quad \text{for } n=1:N\ ; \quad b(\lambda) = a'\,\lambda\ . \tag{21}$$

That is,

$$\mu_{\mathbf{y}}(\lambda) = \prod_{n=1}^{N} \mu_{\mathbf{x}_n}\left(b_n(\lambda)\right) \ . \tag{22}$$

The joint CGF of RVs $\{\mathbf{y}_m\}$, using equation (18), is

$$\chi_{\mathbf{y}}(\lambda) = \ln \mu_{\mathbf{y}}(\lambda) = \sum_{n=1}^{N} \chi_{\mathbf{x}_n}\left(b_n(\lambda)\right) \ . \tag{23}$$

There follows, by reference to equation (21),

$$\frac{\partial \chi_{\mathbf{y}}(\lambda)}{\partial \lambda_m} = \sum_{n=1}^{N} a_{mn}\, \chi'_{\mathbf{x}_n}\left(b_n(\lambda)\right) \quad \text{for } m=1:M \ . \tag{24}$$

Finally, the M simultaneous equations that must be solved for the MD SP $\hat{\lambda}$ are

$$\sum_{n=1}^{N} a_{mn}\, \chi'_{\mathbf{x}_n}\left(b_n(\hat{\lambda})\right) = y_m \quad \text{for } m=1:M \ , \tag{25}$$

where $y = [y_1 \cdots y_M]'$ are the particular values (field point) at which the joint PDF of RVs $\{\mathbf{y}_m\}$ in equation (19) is to be evaluated. The second-order partial derivatives required to obtain the SPA follow from equations (24) and (21) as

$$\frac{\partial^2 \chi_{\mathbf{y}}(\lambda)}{\partial \lambda_m \, \partial \lambda_{\underline{m}}} = \sum_{n=1}^{N} a_{mn}\, a_{\underline{m}n}\, \chi''_{\mathbf{x}_n}\left(b_n(\lambda)\right) \quad \text{for } m,\underline{m}=1:M \ . \tag{26}$$

Once SP $\hat{\lambda}$ is found from equation (25), it can be substituted in equation (26) and the M×M HM can be evaluated at the SP; namely, $\nabla^2 \chi_{\mathbf{y}}(\hat{\lambda})$.

EXTENSION TO DISTORTED RANDOM VARIABLES

Suppose that each independent RV x_n is the result of a different nonlinear transformation according to

$$x_n = h_n(\underline{x}_n) \quad \text{for } n=1:N \ , \tag{27}$$

where RV $\underline{x}_n$ has known first-order PDF $\underline{p}_n(u)$. Then, the first-order MGF of RV x_n is

$$\mu_{x_n}(\lambda) = \overline{\exp(\lambda\, x_n)} = \overline{\exp\left(\lambda\, h_n(\underline{x}_n)\right)} =$$

$$= \int du\ \underline{p}_n(u)\ \exp\left(\lambda\, h_n(u)\right) \quad \text{for } n=1:N \ , \tag{28}$$

from which there follows

$$\mu'_{x_n}(\lambda) = \int du\ \underline{p}_n(u)\ h_n(u)\ \exp\left(\lambda\, h_n(u)\right) \quad \text{for } n=1:N \ . \tag{29}$$

If these 2N integrals in equations (28) and (29) can be evaluated in closed form, then functions

$$\chi'_{x_n}(\lambda) = \frac{\mu'_{x_n}(\lambda)}{\mu_{x_n}(\lambda)} \quad \text{for } n=1:N \tag{30}$$

are available for use in the determination of the SP required in equation (25). Otherwise, numerical integration could be used on these one-dimensional integrals.

PROBABILITY DENSITY FUNCTION OF THE MAXIMUM RANDOM VARIABLE

For M = 2, let scalar RV $\mathbf{y} = \max(z_1, z_2)$. Let $u = [u_1 \; u_2]^T$, $Y = [y \; y]^T$, and real nonlinearities

$$g_1(u) = \delta(y - u_1)\, U(y - u_2) \;, \quad g_2(u) = \delta(y - u_2)\, U(y - u_1) \;. \tag{31}$$

Then, total nonlinearity $g(u) = g_1(u) + g_2(u)$ is real and nonnegative. The average corresponding to total nonlinearity $g(u)$ is

$$a = \int du\; g(u)\; p_z(u) = \int_{-\infty}^{y} du_2\; p_z(y, u_2) + \int_{-\infty}^{y} du_1\; p_z(u_1, y) \;, \tag{32}$$

which is the PDF $p_y(y)$ of scalar RV $\mathbf{y} = \max(z_1, z_2)$ at argument y.

The corresponding gamma functions to $g_1(u)$ and $g_2(u)$ are

$$\gamma_1(\lambda) = \frac{\exp(-\lambda^T Y)}{-\lambda_2} \quad \text{for } \mathrm{Re}(\lambda_2) < 0 \;,\; \lambda_1 \text{ arbitrary;} \tag{33}$$

$$\gamma_2(\lambda) = \frac{\exp(-\lambda^T Y)}{-\lambda_1} \quad \text{for } \mathrm{Re}(\lambda_1) < 0 \;,\; \lambda_2 \text{ arbitrary.} \tag{34}$$

The total gamma function corresponding to nonlinearity $g(u)$ is

$$\gamma(\lambda) = \gamma_1(\lambda) + \gamma_2(\lambda) \quad \text{for } \mathrm{Re}(\lambda_1) < 0 \;\; \underline{\text{and}} \;\; \mathrm{Re}(\lambda_2) < 0 \;. \tag{35}$$

However, notice that combining the two individual gamma functions together has forced a severe restriction on the allowed region in λ-space to investigate for an SP; namely,

$$a = \frac{1}{(i2\pi)^2} \int_{C_\lambda} d\lambda\ \mu_z(\lambda)\ \exp\left(-\lambda_1 y - \lambda_2 y\right)\ \left(\frac{1}{-\lambda_2} + \frac{1}{-\lambda_1}\right)\ , \qquad (36)$$

where component contours C_1 and C_2 of C_λ must <u>both</u> pass to the left of their origins. For $y > 0$, the restrictions to the left-half λ planes keep the real part of the argument of exp( ) positive; therefore, exp( ) is unable to track the rapidly decreasing upper tail of the PDF of **y**. The SP components approach 0- in all $M = 2$ dimensions as threshold y increases, but this is not sufficient to allow development of a rapid decrease of the SPA on the upper tail of the PDF. See figure 10 of reference 2 for an example of this limitation.

On the other hand, for nonlinearity $g_1(u)$ alone, the contribution to average a is

$$a_1 = \frac{1}{(i2\pi)^2} \int_{C^{(1)}} d\lambda\ \mu_z(\lambda)\ \exp(-\lambda_1 y - \lambda_2 y)/(-\lambda_2)\ , \qquad (37)$$

which requires only that the λ_2 contour pass to the left of its origin. Then, the λ_1 component of the SP can move into its right-half plane and yield exponential decay of a_1 with y, even when threshold $y > 0$. Conversely, for average a_2, the λ_2 component of the SP is free to move as needed. In this manner, much better SPAs are obtained, as well as smaller correction terms. The tradeoff is that $M = 2$ SPs must be located, instead of a single SP for equation (36).

In M dimensions, the m-th nonlinearity required to yield the PDF of scalar RV $y = \max(z_1,\ldots,z_M)$ is

$$g_m(u) = \delta(y - u_m) \prod_{\substack{n=1 \\ n\neq m}}^{M} U(y - u_n) \quad \text{for } m=1:M \ , \tag{38}$$

for which the corresponding gamma function is

$$\gamma_m(\lambda) = \exp\left(-y \sum_{n=1}^{M} \lambda_n\right) \Big/ \prod_{\substack{n=1 \\ n\neq m}}^{M} (-\lambda_n) \quad \text{for } m=1:M \ , \tag{39}$$

provided that $\mathrm{Re}(\lambda_n) < 0$ for n=1:M, $n\neq m$, while λ_m is arbitrary. Therefore, the m-th average

$$a_m = \frac{1}{(i2\pi)^M} \int_{C^{(m)}} d\lambda \ \mu_z(\lambda) \exp\left(-y \sum_{n=1}^{M} \lambda_n\right) \Big/ \prod_{\substack{n=1 \\ n\neq m}}^{M} (-\lambda_n) \tag{40}$$

allows for the m-th component contour $C_m^{(m)}$ to move into the right-half λ_m-plane (for each m value in [1,M]) and thereby realize significant exponential decay for a_m, even if threshold $y > 0$. On the other hand, an attempt to minimize the computational effort and use total gamma function

$$\gamma(\lambda) = \sum_{m=1}^{M} \gamma_m(\lambda) \ , \tag{41}$$

as in equation (36) for M = 2, would require that $\mathrm{Re}(\lambda_m) < 0$ for <u>all</u> m=1:M, and thereby severely limit the capability of the resulting SPA, especially for $y > 0$.

JOINT STATISTICS FOR NORMALIZATION

Random vector $\mathbf{x} = [\mathbf{x}_1 \cdots \mathbf{x}_M]'$ has arbitrary joint PDF $p_{\mathbf{x}}$ and joint MGF $\mu_{\mathbf{x}}$. The individual RVs $\{\mathbf{x}_m\}$ can be positive or negative, and can be statistically dependent on each other.

GENERAL NORMALIZER

Scalar RV $\mathbf{y}$ may be statistically dependent on random vector $\mathbf{x}$; however, $\mathbf{y}$ is positive (for the time being). The first-order PDF of $\mathbf{y}$ is $p_{\mathbf{y}}$. The normalizer of interest forms the M ratios

$$\mathbf{z}_m = \frac{\mathbf{x}_m}{\mathbf{y}} \quad \text{for } m=1:M \ . \tag{42}$$

The joint cumulative distribution function (CDF) of random vector $\mathbf{z} = [\mathbf{z}_1 \cdots \mathbf{z}_M]'$ using $\mathbf{y} > 0$ is

$$c_{\mathbf{z}}(z_1,\ldots,z_M) = \text{Prob}(\mathbf{z}_1 < z_1,\ldots,\mathbf{z}_M < z_M) =$$

$$= \text{Prob}(\mathbf{x}_1 < z_1\,\mathbf{y},\ldots,\mathbf{x}_M < z_M\,\mathbf{y}) =$$

$$= \int_0^\infty dy \int_{-\infty}^{z_1 y} dx_1 \cdots \int_{-\infty}^{z_M y} dx_M \; p_{\mathbf{xy}}(x_1,\ldots,x_M,y) =$$

$$= \int_0^\infty dy \; p_{\mathbf{y}}(y) \int_{-\infty}^{z_1 y} dx_1 \cdots \int_{-\infty}^{z_M y} dx_M \; p_{\mathbf{x}}(x_1,\ldots,x_M|\mathbf{y} = y) \ , \tag{43}$$

where $p_{\mathbf{xy}}$ is the joint PDF of $\mathbf{x}$ and $\mathbf{y}$, and $p_{\mathbf{x}}(x|\mathbf{y} = y)$ is the conditional joint PDF of RV $\mathbf{x}$ at argument x, given that scalar RV $\mathbf{y} = y$.

The joint PDF of normalizer vector output z follows from equation (43), upon M partial differentiations, as

$$p_z(z_1,\ldots,z_M) = \int_0^\infty dy\ y^M\ p_y(y)\ p_x(z_1\ y,\ldots,z_M\ y|y = y) = \quad (44)$$

$$= \int_0^\infty dy\ y^M\ p_{xy}(z_1\ y,\ldots,z_M\ y,y)\ . \quad (45)$$

If joint PDF p_{xy} can be determined, a one-dimensional integral must be evaluated in order to obtain joint PDF p_z.

If <u>all</u> the RVs $\{x_m\}$ are statistically independent of scaling y used in normalizer ratios (42), then equation (44) simplifies to

$$p_z(z_1,\ldots,z_M) = \int_0^\infty dy\ y^M\ p_y(y)\ p_x(z_1\ y,\ldots,z_M\ y)\ . \quad (46)$$

This result still requires a one-dimensional integral for its evaluation.

If RV y can take on both positive and negative values, the three integrals in equations (44) through (46) are generalized to the extent that the integrals over y are conducted from $-\infty$ to $+\infty$, and the term y^M is replaced by $|y|^M$. Determination of the conditional PDF in the integrand of equation (44) can be very involved; it may be safer and easier to use the form involving the joint PDF in equation (45). In general, the analytic

difficulty of determining joint PDF p_{xy} prompts us to consider specific forms of normalizers, and attempt to find the joint MGFs of their outputs.

A SPECIFIC NORMALIZER

Let denominator scaling y in normalizer ratios (42) be formed by linear weighting of random vector x according to

$$y = \alpha \sum_{m=1}^{M} x_m + w \ , \tag{47}$$

where scalar RV w is independent of x. For example, w might involve averaging additional RVs $\{x_m\}$, $m > M$, which are independent of $\{x_m\}$, m=1:M. The MGF of scalar RV w is μ_w. Observe that numerators $\{x_m\}$ are statistically dependent on denominator y in normalizers (42) and (47); the one exception is when $\alpha = 0$ in equation (47).

It is presumed that scalar RV y is always positive. Then, upon substitution of equation (47) into equation (42), the joint CDF of normalizer output z is

$$\begin{aligned} C_z(z_1,\ldots,z_M) &= \text{Prob}(x_1 < z_1\, y,\ldots,x_M < z_M\, y) = \\ &= \text{Prob}(z_1\, y - x_1 > 0,\ldots,z_M\, y - x_M > 0) = \\ &= \text{Prob}(v_1 > 0,\ldots,v_M > 0) \ , \end{aligned} \tag{48}$$

where RVs

$$v_m \equiv z_m\, y - x_m = z_m\, \alpha \sum_{\underline{m}=1}^{M} x_{\underline{m}} + z_m\, w - x_m =$$

$$\equiv \sum_{\underline{m}=1}^{M} a_{m\underline{m}}\, x_{\underline{m}} + z_m\, w \quad \text{for } m=1:M \ . \tag{49}$$

Alternatively, with $z = [z_1,\ldots,z_M]'$, the random vector

$$v = [v_1 \cdots v_M]' = A\, x + z\, w \ , \tag{50}$$

where MxM matrix

$$A = A(z,\alpha) = [a_{m\underline{m}}] = \alpha\, \mathrm{diag}(z_1 \cdots z_M)\, \underline{1} - I_M \ , \tag{51}$$

and $\underline{1}$ is an MxM matrix of all ones.

The joint MGF of random vector v is

$$\mu_v(\lambda) = \overline{\exp(\lambda' v)} = \overline{\exp(\lambda' A x + \lambda' z w)} =$$

$$= \overline{\exp(x'\, b(\lambda))}\ \overline{\exp(\lambda' z w)} = \mu_x(b(\lambda))\ \mu_w(z'\lambda) \ , \tag{52}$$

using the independence of random vector x and random scalar w, and defining Mx1 vector

$$b(\lambda) = A'\lambda \ ; \quad b_m(\lambda) = \sum_{\underline{m}=1}^{M} a_{\underline{m}m}\, \lambda_{\underline{m}} \quad \text{for } m=1:M \ . \tag{53}$$

It is also presumed above that $b(\lambda) \in \mathrm{ROA}(\mu_x)$ and that $z'\lambda \in \mathrm{ROA}(\mu_w)$. Finally, the joint CDF of normalizer output z follows from equations (48) and (52) as (see reference 2)

$$c_z(z) = e_v(0) = \frac{1}{(i2\pi)^M} \int_{C_r} \cdots \int d\lambda \; \mu_v(\lambda) \Big/ \prod_{m=1}^{M} (\lambda_m) =$$

$$= \frac{1}{(i2\pi)^M} \int_{C_r} \cdots \int d\lambda \; \mu_x(b(\lambda)) \; \mu_w(z'\lambda) \Big/ \prod_{m=1}^{M} (\lambda_m) \; . \qquad (54)$$

In the special case where RVs $\{x_m\}$ are independent of each other, but not necessarily identically distributed, the joint MGF required in equation (54) becomes

$$\mu_x(b(\lambda)) = \prod_{m=1}^{M} \mu_{x_m}\left(b_m(\lambda)\right) \qquad (55)$$

in terms of the individual MGFs $\{\mu_{x_m}\}$ of RVs $\{x_m\}$. Notice that the statistical dependence of all the RVs $\{x_n\}$, n=1:N, on denominator RV y in equations (42) and (47) is fully accounted for in equations (54) and (55), basically through MxM matrix A in equation (51) and Mx1 vector $b(\lambda)$ in equation (53).

For general RVs $\{x_m\}$, if the additional RV w in normalizer denominator y in equation (47) is zero, then $\mu_w(u) = 1$ for all u; this simplifies the evaluation of equation (54). However, observe that $A = A(z,\alpha)$, leading to

$$b(\lambda) = A(z,\alpha)' \; \lambda = b(z,\alpha,\lambda) \; , \qquad (56)$$

through equation (53). On the other hand, if w is a nonzero constant, then its MGF is simply $\mu_w(u) = \exp(wu)$, thereby leading to the term $\exp(wz'\lambda)$ in equation (54) for $\mu_w(z'\lambda)$.

Alternatively, if scale factor $\alpha = 0$ in equation (47), then matrix A in equation (51) is $-I_M$, and equation (53) yields $b(\lambda) = -\lambda$. Then, equation (54) yields joint CDF

$$c_z(z) = \frac{1}{(i2\pi)^M} \int_{C_r} \cdots \int d\lambda \; \mu_x(-\lambda) \; \mu_w(z'\lambda) \Big/ \prod_{m=1}^{M} (\lambda_m) \; . \tag{57}$$

If RVs $\{x_m\}$ are independent of each other, there follows

$$\mu_x(-\lambda) = \prod_{m=1}^{M} \mu_{x_m}(-\lambda_m) \; . \tag{58}$$

ALTERNATIVE PDF APPROACH

In equation (49), define RVs

$$u_m = \sum_{\underline{m}=1}^{M} a_{m\underline{m}} \; x_{\underline{m}} \quad \text{for } m=1:M \; . \tag{59}$$

Since this is a one-to-one transformation, the joint PDF p_u can be immediately found from joint PDF p_x. Then, for $w > 0$, equations (48) and (49) take the form

$$c_z(z) = \text{Prob}(u_1 > -z_1 \; w, \ldots, u_M > -z_M \; w) =$$

$$= \int_0^{\infty} dw \; p_w(w) \; e_u(-z_1 \; w, \ldots, -z_M \; w) \; , \tag{60}$$

using the independence of vector u and RV w, as well as the joint exceedance distribution function e_u. The corresponding joint PDF is

$$p_z(z) = \int_0^\infty dw\ w^M\ p_w(w)\ p_u(-z_1\ w,\ldots,-z_M\ w)\ , \qquad (61)$$

which agrees with equation (46) under a re-identification of variables.

IMPROVING THE ACCURACY OF THE M-DIMENSIONAL SADDLEPOINT APPROXIMATION BY MEANS OF GAUSS QUADRATURE

Let $\mathbf{x} = [x_1 \cdots x_M]'$ be an MD RV with joint MGF $\mu_{\mathbf{x}}(\lambda) = \overline{\exp(\lambda'\mathbf{x})}$, where vector $\lambda = [\lambda_1 \cdots \lambda_M]'$. Then, the joint PDF of $\mathbf{x}$ at MD field point $x = [x_1 \cdots x_M]'$ is given by the M-th order integral

$$p_{\mathbf{x}}(x) = \frac{1}{(i2\pi)^M} \int_C d\lambda \exp(-\lambda'x)\ \mu_{\mathbf{x}}(\lambda) \ , \tag{62}$$

where BC C is parallel to the imaginary axis in each of the M dimensions. The joint CGF of $\mathbf{x}$ is $\chi_{\mathbf{x}}(\lambda) = \ln \mu_{\mathbf{x}}(\lambda)$. The MD SP of the integrand of equation (62) is that real point $\hat{\lambda} = \hat{\lambda}(x)$ nearest the origin where all M partial derivatives satisfy

$$\left.\frac{\partial \chi_{\mathbf{x}}(\lambda)}{\partial \lambda_m}\right|_{\hat{\lambda}} = x_m \quad \text{for } m=1:M \ . \tag{63}$$

When the contour C is moved in M dimensions so as to go through the SP, and the change of variable

$$\lambda = \hat{\lambda} + i\ t \ , \qquad t = [t_1 \cdots t_M]' \tag{64}$$

is made, equation (62) becomes

$$p_{\mathbf{x}}(x) = (2\pi)^{-M} \exp(-\hat{\lambda}'x) \int dt \exp(-it'x)\ \mu_{\mathbf{x}}(\hat{\lambda}+it) \ , \tag{65}$$

where the new contour passes through the SP of the integrand of equation (65), which is now at $t = 0$.

The logarithm of the integrand of equation (65) can be expanded in a power series about the origin according to

$$\ln\{\exp(-it'x)\ \mu_x(\hat{\lambda}+it)\} = -it'x + \chi_x(\hat{\lambda}+it) =$$

$$\cong -it'x + \chi_x(\hat{\lambda}) + it'\ \nabla\chi_x(\hat{\lambda}) - \frac{1}{2}\ t'\ H_x(\hat{\lambda})\ t\ , \qquad (66)$$

where M×M matrix

$$H_x(\lambda) \equiv \left[\frac{\partial^2 \chi_x(\lambda)}{\partial\lambda_m\ \partial\lambda_{\underline{m}}}\right]\ ,\quad m,\underline{m}=1:M\ , \qquad (67)$$

is symmetric in m and $\underline{m}$ for all λ. Thus, using equations (66) and (63), the integrand of equation (65) can be approximated as

$$\exp(-it'x)\ \mu_x(\hat{\lambda}+it) \cong \exp\left(\chi_x(\hat{\lambda}) - \frac{1}{2}\ t'\ H_x(\hat{\lambda})\ t\right) \qquad (68)$$

for small $|t|$. If this approximation is now extrapolated to <u>all</u> t and substituted in equation (65), there follows the usual SP (or tilted Edgeworth) approximation in M dimensions:

$$p_x(x) \cong \frac{\exp\left(\chi_x(\hat{\lambda})-\hat{\lambda}'x\right)}{(2\pi)^M} \int_{-\infty}^{\infty} dt\ \exp\left(-\frac{1}{2}\ t'\ H_x(\hat{\lambda})\ t\right) =$$

$$= \frac{\exp\left(\chi_x(\hat{\lambda})-\hat{\lambda}'x\right)}{(2\pi)^{M/2}\ \det\left(H_x(\hat{\lambda})\right)^{\frac{1}{2}}}\ ;\quad \hat{\lambda} = \hat{\lambda}(x)\ . \qquad (69)$$

The extrapolation of approximation (66) to all t was done for purposes of analytically evaluating the MD integral in equation (65). If, instead, the MD change of variable

$$t = H_x(\hat{\lambda})^{-\frac{1}{2}}\, s \quad (s \text{ is } M\times 1) \tag{70}$$

is made in equation (65), there follows the exact PDF form

$$p_x(x) = \frac{\exp\left(\chi_x(\hat{\lambda})-\hat{\lambda}'x\right)}{(2\pi)^{M/2}\,\det\left(H_x(\hat{\lambda})\right)^{\frac{1}{2}}}\, I\ , \tag{71}$$

where the multiplicative M-fold integral is

$$I = (2\pi)^{-M/2}\int_{-\infty}^{\infty} ds\, \exp\left(-i\ s'\ H_x(\hat{\lambda})^{-\frac{1}{2}}\ x\right)\ \frac{\mu_x\left(\hat{\lambda} + i\ H_x(\hat{\lambda})^{-\frac{1}{2}}\ s\right)}{\mu_x(\hat{\lambda})}\ . \tag{72}$$

The integrand of this latter integral behaves as exp(-s's/2) near the SP at the origin of s space; this leads to the earlier approximation above, namely, $I \cong 1$.

For one-dimensional integrals in x that behave as $\exp(-a\ x^2)$ near the origin, the use of Gauss quadrature (reference 3, page 924) suggests itself as a good candidate for high accuracy with few integrand evaluations. This quadrature approach can be readily extended to M dimensions by simply repeating the one-dimensional rule for each of the M dimensions under investigation. This is consistent with the fact that integral (72) has been transformed, and for which the integrand behaves similarly in all dimensions, namely as $\exp(-s_m^2/2)$ in the m-th dimension, at least near the SP at the origin s = 0. The numerical question to answer in practice is: how close is integral I in equation (72) to the value 1?

The basic problem with the quadrature approach in high numbers of dimensions is that if S samples are taken per dimension of the integrand, the total number of samples required in M dimensions is S^M. For example, just S = 2 samples per dimension requires, in M = 16 dimensions, 66,000 evaluations of a complex integrand which is a function of M variables. This can be a very time-consuming task and require considerable storage, depending on how large M gets.

Define the residual integrand of equation (72) as the actual integrand multiplied by exp(s's/2); this residual integrand is presumed to be 1 for all s under the extrapolation assumption leading to equation (69). On the other hand, Gauss quadrature, using S samples in a particular dimension, is effectively fitting the residual integrand with a polynomial of order 2S-1 in that variable, which affords a rather high-order fit for few samples. In M dimensions, it means that the Gauss quadrature procedure is <u>exact</u> for residual integrands of the form

$$\sum_{p_m} a(p_1,\ldots,p_M)\ s_1^{p_1} \cdots s_M^{p_M} \qquad (73)$$

for any constants $\{a(p_1,\ldots,p_M)\}$, where integer $p_m \leq 2S-1$ for all m=1:M. This potentially high order of accuracy makes the use of Gauss quadrature on equation (72) a strong candidate for increased accuracy of the SPA if the large number of integrand evaluations can be tolerated.

FIRST-ORDER CORRECTION TERM TO SADDLEPOINT APPROXIMATION

The first-order (FO) correction term to the SPA (or tilted Edgeworth approximation), that is, the O(1/N) term, is given in reference 4, page 180. To develop it fully for these purposes, the following definitions are employed here. For RV **x** with MD CGF $\chi_{\mathbf{x}}$, and MD field point x of interest with corresponding SP $\hat{\lambda} = \hat{\lambda}(x)$, let j,k,ℓ,m = 1:M, and define

$$\chi_{\ell m} = \left. \frac{\partial^2 \chi_{\mathbf{x}}(\lambda)}{\partial \lambda_\ell \, \partial \lambda_m} \right|_{\hat{\lambda}} , \tag{74}$$

$$\chi_{k\ell m} = \left. \frac{\partial^3 \chi_{\mathbf{x}}(\lambda)}{\partial \lambda_k \, \partial \lambda_\ell \, \partial \lambda_m} \right|_{\hat{\lambda}} , \tag{75}$$

$$\chi_{jk\ell m} = \left. \frac{\partial^4 \chi_{\mathbf{x}}(\lambda)}{\partial \lambda_j \, \partial \lambda_k \, \partial \lambda_\ell \, \partial \lambda_m} \right|_{\hat{\lambda}} . \tag{76}$$

Also, define the two M×M matrices

$$H = [\chi_{\ell m}] , \qquad T = [T_{\ell m}] = H^{-1} . \tag{77}$$

H is the HM of the MD CGF $\chi_{\mathbf{x}}$ of RV **x**, evaluated at the SP $\hat{\lambda}$.

Then, the FO-corrected joint PDF at MD field point x is

$$p_1(x) = p_0(x) \, [1 + c_4 + c_{3a} + c_{3b}] , \tag{78}$$

where

$$p_0(x) = \frac{\exp\left(\chi_x(\hat{\lambda}) - \hat{\lambda}' x\right)}{\left((2\pi)^M \det(H)\right)^{\frac{1}{2}}} \tag{79}$$

is the usual (zeroth-order) SPA, and parameters

$$c_4 = \frac{1}{8} \sum_{jk\lambda m} \chi_{jk\lambda m} T_{jk} T_{\lambda m} , \tag{80}$$

$$c_{3a} = -\frac{1}{8} \sum_{k\lambda m} \sum_{\underline{k}\underline{\lambda}\underline{m}} \chi_{k\lambda m} \chi_{\underline{k}\underline{\lambda}\underline{m}} T_{k\lambda} T_{m\underline{k}} T_{\underline{\lambda}\underline{m}} , \tag{81}$$

$$c_{3b} = -\frac{1}{12} \sum_{k\lambda m} \sum_{\underline{k}\underline{\lambda}\underline{m}} \chi_{k\lambda m} \chi_{\underline{k}\underline{\lambda}\underline{m}} T_{k\underline{k}} T_{\lambda\underline{\lambda}} T_{m\underline{m}} . \tag{82}$$

All three terms (c_4, c_{3a}, c_{3b}) are O(1/N) if N independent RVs have been added to form RV **x**. However, the results above hold for any RV **x**, no matter how formed. The two component terms, c_{3a} and c_{3b}, cannot be combined together, in general.

APPLICATION TO WEIGHTED SUMS OF RANDOM VARIABLES

Let N-dimensional RV $\mathbf{w} = [w_1 \cdots w_N]'$ be a collection of N <u>independent</u> RVs, with possibly different statistics for each RV w_n, n=1:N. In particular, let individual RV w_n have FO MGF μ_{w_n} and FO CGF χ_{w_n} for n=1:N. The MD RV **x** of interest is formed according to linear transformation

$$\mathbf{x} = a\,\mathbf{w} , \quad a = \left[a_{mn}\right] . \tag{83}$$

Arbitrary array a is MxN, $M \leq N$, and of rank M. Then, RV **x** is

composed of M dependent RVs with joint MGF

$$\mu_{\mathbf{x}}(\lambda) = \overline{\exp(\lambda' \mathbf{x})} = \overline{\exp(\lambda' \mathbf{a} \mathbf{w})} = \overline{\exp(\mathbf{b}(\lambda)' \mathbf{w})} , \tag{84}$$

where Nx1 vector

$$\mathbf{b}(\lambda) = \mathbf{a}' \lambda = \left[b_1(\lambda) \cdots b_N(\lambda) \right]' , \tag{85}$$

with elements

$$b_n(\lambda) = \sum_{m=1}^{M} a_{mn} \lambda_m \quad \text{for } n=1:N . \tag{86}$$

Notice that

$$\frac{\partial b_n(\lambda)}{\partial \lambda_m} = a_{mn} \quad \text{for } m=1:M , \; n=1:N . \tag{87}$$

Using the independence of RVs $\{w_n\}$, there follows for the joint MGF of $\mathbf{x}$ from equations (84) through (86),

$$\mu_{\mathbf{x}}(\lambda) = \prod_{n=1}^{N} \overline{\exp\left(b_n(\lambda) \, w_n\right)} = \prod_{n=1}^{N} \mu_{w_n}\left(b_n(\lambda)\right) . \tag{88}$$

The joint CGF of RV $\mathbf{x}$ is then

$$\chi_{\mathbf{x}}(\lambda) = \sum_{n=1}^{N} \chi_{w_n}\left(b_n(\lambda)\right) . \tag{89}$$

There follows, upon use of equation (87),

$$\frac{\partial \chi_{\mathbf{x}}(\lambda)}{\partial \lambda_m} = \sum_{n=1}^{N} a_{mn} \, \chi'_{w_n}\left(b_n(\lambda)\right) \quad \text{for } m=1:M , \tag{90}$$

$$\frac{\partial^2 \chi_{\mathfrak{k}}(\lambda)}{\partial \lambda_{\mathfrak{k}} \, \partial \lambda_m} = \sum_{n=1}^{N} a_{\mathfrak{k}n} \, a_{mn} \, \chi''_{w_n}\left(b_n(\lambda)\right) \quad \text{for } \mathfrak{k}=1:M \ , \quad m=1:M \ . \tag{91}$$

Therefore, the M simultaneous equations that must be solved for the SP $\hat{\lambda} = \hat{\lambda}(x)$ at MD field point x are

$$\sum_{n=1}^{N} a_{mn} \, \chi'_{w_n}\left(b_n(\hat{\lambda})\right) = x_m \quad \text{for } m=1:M \ . \tag{92}$$

In order to develop explicitly the component terms in equations (80) through (82), define, for n=1:N,

$$\chi_{2n} = \chi''_{w_n}\left(b_n(\hat{\lambda})\right) \ , \quad \chi_{3n} = \chi'''_{w_n}\left(b_n(\hat{\lambda})\right) \ , \quad \chi_{4n} = \chi''''_{w_n}\left(b_n(\hat{\lambda})\right) \ . \tag{93}$$

These 3N quantities only need to be computed at the SP $\hat{\lambda}$. The use of equations (93) and (74) in equation (91) yields

$$\chi_{\mathfrak{k}m} = \sum_{n=1}^{N} a_{\mathfrak{k}n} \, a_{mn} \, \chi_{2n} \quad \text{for } \mathfrak{k}=1:M \ , \quad m=1:M \ . \tag{94}$$

In a similar fashion, there follows from equations (75), (76), and (93), for j,k,ℓ,m = 1:M,

$$\chi_{k\mathfrak{k}m} = \sum_{n=1}^{N} a_{kn} \, a_{\mathfrak{k}n} \, a_{mn} \, \chi_{3n} \ , \tag{95}$$

$$\chi_{jk\mathfrak{k}m} = \sum_{n=1}^{N} a_{jn} \, a_{kn} \, a_{\mathfrak{k}n} \, a_{mn} \, \chi_{4n} \ . \tag{96}$$

The three component terms in equations (80) - (82) can now be determined. From equations (80) and (96), there follows

$$c_4 = \frac{1}{8} \sum_{jk\lambda m} \sum_{n} a_{jn}\, a_{kn}\, a_{\lambda n}\, a_{mn}\, \chi_{4n}\, T_{jk}\, T_{\lambda m} =$$

$$= \frac{1}{8} \sum_{n=1}^{N} \chi_{4n} \left(\sum_{jk=1}^{M} a_{jn}\, T_{jk}\, a_{kn} \right)^2 . \qquad (97)$$

Define the N×N matrix $J = [J_{n\underline{n}}]$ with elements

$$J_{n\underline{n}} = \sum_{\lambda m=1}^{M} a_{\lambda n}\, T_{\lambda m}\, a_{m\underline{n}} \quad \text{for } n,\underline{n} = 1{:}N . \qquad (98)$$

Then,

$$J = a'\, T\, a , \qquad (99)$$

and equation (97) can be expressed as

$$c_4 = \frac{1}{8} \sum_{n=1}^{N} \chi_{4n}\, J_{nn}^2 . \qquad (100)$$

This result uses only the diagonal of N×N matrix J.

Also, from equations (81) and (95), there follows

$$c_{3a} = -\frac{1}{8} \sum_{k\lambda m} \sum_{\underline{k}\,\underline{\lambda}\,\underline{m}} \sum_{n} a_{kn}\, a_{\lambda n}\, a_{mn}\, \chi_{3n} \sum_{\underline{n}} a_{\underline{k}\underline{n}}\, a_{\underline{\lambda}\underline{n}}\, a_{\underline{m}\underline{n}}\, \chi_{3\underline{n}} \, *$$

$$T_{k\lambda}\, T_{m\underline{k}}\, T_{\underline{\lambda}\underline{m}} = -\frac{1}{8} \sum_{n\underline{n}=1}^{N} \chi_{3n}\, \chi_{3\underline{n}}\, J_{nn}\, J_{n\underline{n}}\, J_{\underline{n}\underline{n}} . \qquad (101)$$

Define N×1 vector $j = [j_1 \cdots j_N]'$ with elements

$$j_n = \chi_{3n}\, J_{nn} \quad \text{for } n=1{:}N . \qquad (102)$$

Then, equation (101) can be expressed compactly as

$$c_{3a} = -\frac{1}{8} \sum_{n\underline{n}=1}^{N} j_n \, J_{n\underline{n}} \, j_{\underline{n}} = -\frac{1}{8} \, j' \, J \, j \; . \qquad (103)$$

The final component term is, from equations (82) and (95),

$$c_{3b} = -\frac{1}{12} \sum_{k\underline{k}m} \sum_{\underline{k}\underline{k}\underline{m}} \sum_{n} a_{kn} \, a_{\underline{k}n} \, a_{mn} \, \chi_{3n} \sum_{\underline{n}} a_{k\underline{n}} \, a_{\underline{k}\underline{n}} \, a_{m\underline{n}} \, \chi_{3\underline{n}} \, *$$

$$T_{k\underline{k}} \, T_{\underline{k}\underline{k}} \, T_{m\underline{m}} = -\frac{1}{12} \sum_{n\underline{n}=1}^{N} \chi_{3n} \, \chi_{3\underline{n}} \, J_{n\underline{n}}^{3} \; . \qquad (104)$$

The total correction term of O(1/N) required for $p_1(x)$ in equation (78) is given by the sum of components (100), (103), and (104). The inputs required here are MxN matrix a, individual FO CGFs $\{\chi_{w_n}\}$ of RVs $\{w_n\}$ for n=1:N, and field point $x = [x_1 \cdots x_M]'$.

The MATLAB code for accomplishing all of these matrix manipulations is very compact once the Nx1 chi-vectors in equation (93) are available:

```
b=a'*lambda;
H=a.*repmat(chi2',M,1)*a';
J=a'/H*a;
Jd=diag(J);
j=chi3.*Jd;
c4=chi4'*(Jd.^2)/8;
c3a=-j'*J*j/8;
c3b=-chi3'*(J.^3)*chi3/12;
```

EXACT M-DIMENSIONAL PROBABILITY DENSITY FUNCTIONS FOR SPECIAL TRANSFORMATIONS

Let $\mathbf{x}$ be a column vector of M real RVs with joint PDF $p_{\mathbf{x}}$. Also, let the real, nonrandom MxM matrix A be of rank M. Then, the joint PDF $p_{\mathbf{y}}$ of the column vector of M real RVs $\mathbf{y} = A\,\mathbf{x}$ can be determined immediately as

$$p_{\mathbf{y}}(y) = p_{\mathbf{x}}\left(x = A^{-1}\,y\right) \Big/ |\det(A)| \ , \tag{105}$$

where $y = [y_1 \ \ldots \ y_M]'$ is the nonrandom field point of interest. It is desired to extend this exact result for the PDF of RV $\mathbf{y}$ to matrices A of size MxN of rank M, where N may be (much) larger than M. This extension is not achievable for all possible A matrices, but can be accomplished to yield exact PDFs of RV $\mathbf{y}$ for linear transformations A of a particular form.

Let V(1),...,V(M) be arbitrary Mx1 real column vectors that are linearly independent of each other. Form the MxN matrix

$$A \equiv \left[d_1\ V(m(1)) \quad d_2\ V(m(2)) \quad \cdots \quad d_N\ V(m(N))\right] \ , \tag{106}$$

where $N \geq M$ and $d_1, d_2, \ldots, d_N$ are N arbitrary real scalars, while m(1), m(2),..., m(N) are N arbitrary integers in the range [1,M]. To make the rank of A equal to M, every vector V(1) through V(M) must appear at least once in matrix A in equation (106). Alternatively stated, the set of N integers m(1), m(2), ..., m(N) must include every integer in the range [1,M] at least once.

Let $\mathbf{x} = [x_1 \cdots x_N]'$, where the N scalar RVs $\{x_n\}$ are statistically independent of each other but are not necessarily identically distributed. Consider the Mx1 RV obtained by the linear transformation

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} = \sum_{n=1}^{N} d_n\, V(m(n))\, x_n \equiv [y_1 \cdots y_M]' \ , \tag{107}$$

where explicit use of equation (106) has been made. Now,

$$\begin{aligned} &\text{let } S_1 \text{ denote the set of n values where } m(n) = 1; \\ &\vdots \\ &\text{let } S_M \text{ denote the set of n values where } m(n) = M. \end{aligned} \tag{108}$$

Then, equation (107) can be expressed as

$$y = V(1) \sum_{n\varepsilon S_1} d_n\, x_n + \cdots + V(M) \sum_{n\varepsilon S_M} d_n\, x_n = \sum_{m=1}^{M} V(m)\, w_m \ , \tag{109}$$

where the M RVs $\{w_m\}$ are defined as

$$w_m = \sum_{n\varepsilon S_m} d_n\, x_n \quad \text{for } m=1:M \ . \tag{110}$$

The M scalar RVs $\{w_m\}$ are statistically independent of each other because sets $S_1,\ldots,S_M$, defined in equation (108), are disjoint and non-empty; therefore, no common RVs of set $\{x_n\}$ can appear in two different RVs of set $\{w_m\}$. The FO characteristic function (CF) of the m-th RV w_m follows immediately from equation (110) as

$$f_{w_m}(\xi) = \prod_{n\varepsilon S_m} f_{x_n}(d_n \xi) \quad \text{for } m=1:M \ , \tag{111}$$

where f_{x_n} is the CF of RV x_n, n=1:N. The one-dimensional Fourier transform of CF f_{w_m} yields the FO PDF p_{w_m} of RV w_m for m=1:M.

Express Mx1 column vector V(m) in terms of its components as

$$V(m) = [v_1(m) \cdots v_M(m)]' \quad \text{for } m=1:M \ , \tag{112}$$

and let Mx1 column vector $w \equiv [w_1 \cdots w_M]'$. Also, let MxM matrix

$$V = [V(1) \cdots V(M)] = \begin{bmatrix} v_1(1) & \cdots & v_1(M) \\ \vdots & & \vdots \\ v_M(1) & \cdots & v_M(M) \end{bmatrix} . \tag{113}$$

Then, equation (109) can be expressed as

$$\mathbf{y} = V \mathbf{w} \ , \quad \text{or} \quad \mathbf{w} = V^{-1} \mathbf{y} \equiv B \mathbf{y} \ ; \tag{114}$$

that is, there is a one-to-one transformation

$$w_m = \sum_{\underline{m}=1}^{M} b_{m\underline{m}} \, y_{\underline{m}} \quad \text{for } m=1:M \ , \quad B = [b_{m\underline{m}}] \ . \tag{115}$$

Now, guided by result (105), the joint M-th order PDF of RV $\mathbf{y}$ in equation (114) is immediately given at field point y by

$$p_y(y) = p_y(y_1,\ldots,y_M) = \frac{p_w\left(V^{-1} \, y\right)}{|\det(V)|} =$$

$$= \frac{1}{|\det(V)|} \prod_{m=1}^{M} p_{w_m}\left(b_{m1} \, y_1 + \cdots + b_{mM} \, y_M\right) \tag{116}$$

upon use of the independence of the M RVs $\{w_m\}$ defined in equation (110).

In order to utilize equation (116), it is necessary that each FO PDF $p_{w_m}(w)$ be capable of evaluation at arbitrary argument w; but this quantity is available by means of a one-dimensional Fourier transform of equation (111):

$$p_{w_m}(w) = \frac{1}{2\pi} \int d\xi \, \exp(-iw\xi) \, f_{w_m}(\xi) \quad \text{for } m=1:M \; . \tag{117}$$

For a given or specified set of values $y = (y_1,\ldots,y_M)$ in equation (116), argument $w_m \equiv b_{m1} y_1 + \cdots + b_{mM} y_M$ for PDF p_{w_m} can be computed, and single integral (117) can be evaluated for this particular w_m value. This basic integral must be redone for each m in the range m=1:M for use in equation (116). An exact value for the PDF $p_y(y)$ of RV y in equation (107) can be obtained in this fashion.

SPECIAL CASE

Let the N integers m(1), ..., m(N) in equation (106) consist of K 1s followed by K 2s ... followed by K Ms, for a total of N = KM integers. Then, set S_m in equation (108) is the set of integers

$$S_m = \{1+(m-1)K,\ldots, mK\} \quad \text{for } m=1:M \; . \tag{118}$$

Use of this relation in equation (111) yields CF

$$f_{w_m}(\xi) = \prod_{n=1+mK-K}^{mK} f_{x_n}(d_n\xi) \quad \text{for } m=1:M \; . \tag{119}$$

Furthermore, specialize to the case of independent identically distributed RVs $\{x_n\}$ with an exponential PDF with unit mean; that is, their common CF is $f_{x_n}(\xi) = (1 - i\xi)^{-1}$ for n=1:N. Then, equation (119) yields the explicit CF relation

$$f_{w_m}(\xi) = \left(\prod_{n=1+mK-K}^{mK} \left(1 - i\xi d_n\right) \right)^{-1} \quad \text{for } m=1:M \ . \tag{120}$$

This result must now be subjected to integral (117) in order to determine the FO PDFs of RVs $\{w_m\}$.

As a further specialization, consider sequence $d_n = 1$ for n=1:N. Then, equation (120) yields CF $f_{w_m}(\xi) = (1 - i\xi)^{-K}$ for all m=1:M, and equation (117) yields closed-form PDFs

$$p_{w_m}(w) = \frac{w^{K-1} \exp(-w)}{(K-1)!} U(w) \quad \text{for } m=1:M \ , \tag{121}$$

where U is the unit-step function. This result can be applied in equation (116) for the joint PDF $p_y(y)$ upon use of $B = V^{-1}$.

Alternatively, consider the complete scalar sequence $\{d_n\}$ to be composed of M groups of K scalars each. Let each group contain J (= K/2) +1s and J -1s, K even. Then, equation (120) yields CF $f_{w_m}(\xi) = (1 + \xi^2)^{-K/2} = (1 + \xi^2)^{-J}$ for all m=1:M, and equation (117) yields FO PDFs

$$p_{w_m}(w) = \frac{\exp(-|w|)}{2^{2J-1}} \sum_{j=0}^{J-1} \binom{2J-2-j}{J-1} \frac{(2|w|)^j}{j!} \quad \text{for } m=1:M \tag{122}$$

for all w. Again, direct application in equation (116) gives exact results for joint PDF $p_y(y)$ at any argument y.

SUMMARY

The dominant saddlepoint in the M-dimensional region of analyticity of the original moment generating function is located on the real axes of λ space, if it exists at all. The numerical search for this saddlepoint is alleviated by the fact that the real integrand of interest has positive curvature (a bowl-like behavior) in the M real dimensions with a single minimum in the region of analyticity. The first-order correction term to the standard saddlepoint approximation requires the calculation of numerous fourth-order partial derivatives of the joint cumulant generating function of the random variables of interest; however, these calculations can be limited to the saddlepoint location alone.

APPENDIX A — BRANCH POINT EXAMPLE

Consider the one-dimensional PDF and MGF pair:

$$p(u) = \frac{u^{\nu} \exp(-u)}{\Gamma(\nu+1)} \quad \text{for } u > 0 , \quad \nu > -1 ;$$

$$\mu(\lambda) = (1 - \lambda)^{-\nu-1} \quad \text{for } \lambda_r < 1 . \tag{A-1}$$

$\mu(\lambda_r)$ is positive real for $\lambda_r < 1$. The AC of $\mu(\lambda)$ has a BP at at $\lambda = 1$ when ν is not an integer. Use this AC of $\mu(\lambda)$ to get the PDF expression

$$p(u) = \frac{1}{i2\pi} \int_C d\lambda \, \exp(-u\lambda)/(1-\lambda)^{\nu+1} , \tag{A-2}$$

where BC C passes to the left of $\lambda = 1$.

Let $u > 0$. Also, let $-1 < \nu < 0$ for now. Wrap contour C around the BP at $\lambda = 1$, making a keyhole contour centered on the positive-real λ-axis; this equivalent contour is the steepest descent contour. The infinitely small circular contour around $\lambda = 1$ yields a zero contribution in the limit because $\nu < 0$. Let $\lambda = 1 + r \exp(i\phi)$; $1 - \lambda = -r \exp(i\phi) = r \exp(i\phi - i\pi)$.

When $\phi = \pi$, $1 - \lambda = r > 0$ and $(1-\lambda)^{\nu+1}$ is positive real, as required. On the upper line integral, $\phi = 0$, giving $(1-\lambda)^{\nu+1} = r^{\nu+1} \exp[-i\pi(\nu+1)]$; on the lower line integral, $\phi = 2\pi$, giving $(1-\lambda)^{\nu+1} = r^{\nu+1} \exp[i\pi(\nu+1)]$. Substitution into equation (A-2) yields, for $u > 0$,

$$p(u) = \frac{1}{i2\pi} \int_{0+}^{+\infty} dr \; \frac{\exp[-u(1+r)]}{r^{\nu+1} \exp[-i\pi(\nu+1)]} + \text{complex conjugate}$$

$$= \frac{\exp(-u)}{\pi} \, \mathrm{Re}\left[\frac{\exp[i\pi(\nu+1)]}{i} \, \frac{\Gamma(-\nu)}{u^{-\nu}} \right]$$

$$= u^{\nu} \exp(-u) \, \frac{\sin[\pi(\nu+1)]}{\pi} \, \Gamma(-\nu) = \frac{u^{\nu} \exp(-u)}{\Gamma(\nu+1)} \qquad \text{(A-3)}$$

upon use of reference 3, equation 6.1.17. Now, use AC on ν to extend this result to all $\nu > -1$. This is an exact approach and result; it returns equation (A-1), as expected.

On the other hand, the SPA uses the SP of the integrand of equation (A-2); namely, $\Psi(\lambda,u) = \exp(-u\lambda)/(1-\lambda)^{\nu+1}$ at location $\hat{\lambda}(u) = 1 - (\nu+1)/u$ for $u > 0$. The resulting SPA is

$$p_o(u) = \frac{\exp(\nu+1)}{(2\pi)^{\frac{1}{2}} \, (\nu+1)^{\nu+\frac{1}{2}}} \, u^{\nu} \exp(-u) \quad \text{for } u > 0 \, , \; \nu > -1 \, . \qquad \text{(A-4)}$$

This form is identical to PDF $p(u)$ in equations (A-1) and (A-3) except for a scale factor. The ratio $p_o(u)/p(u)$ is independent of u and approaches 1+ as ν increases; the ratio is $e/\sqrt{(2\pi)} = 1.0844$ at $\nu = 0$, and is $e^2/(4\sqrt{\pi}) = 1.0422$ at $\nu = 1$.

APPENDIX B — HESSIAN MATRIX OF JOINT CUMULANT GENERATING FUNCTION $\chi(\lambda)$ IS POSITIVE DEFINITE FOR λ REAL AND IN R_μ

Vector field point $z = [z_1 \cdots z_M]^T$ is real and MD. Let $p(z)$ be the joint PDF of MD RV z at field point z; thus, $p(z)$ is real, nonnegative, and has unit volume in MD z space.

Vector $\lambda = [\lambda_1 \cdots \lambda_M]^T$ is complex and MD. The joint MGF $\mu(\lambda)$ corresponding to joint PDF $p(z)$ is given by Laplace transform and expectation

$$\mu(\lambda) = \int dz \, \exp(\lambda^T z) \, p(z) = E\{\exp(\lambda^T \mathbf{z})\} \; . \tag{B-1}$$

Henceforth, it is presumed that $\lambda \in \mathrm{ROA}(\mu)$; that is, $\lambda_r \in R_\mu$. Define

$$\frac{\partial}{\partial \lambda_m} \mu(\lambda) = \mu_m(\lambda) \quad \text{for } m=1:M \; . \tag{B-2}$$

Then, the joint CGF $\chi(\lambda)$ of RV z satisfies the relations

$$\chi(\lambda) = \ln \mu(\lambda) \; ,$$

$$\frac{\partial}{\partial \lambda_m} \chi(\lambda) = \frac{\mu_m(\lambda)}{\mu(\lambda)} \quad \text{for } m=1:M \; ,$$

$$\frac{\partial^2}{\partial \lambda_m \, \partial \lambda_{\underline{m}}} \chi(\lambda) = \frac{\mu_{m\underline{m}}(\lambda)}{\mu(\lambda)} - \frac{\mu_m(\lambda)}{\mu(\lambda)} \frac{\mu_{\underline{m}}(\lambda)}{\mu(\lambda)} \quad \text{for } m,\underline{m}=1:M \; . \tag{B-3}$$

The M×M HM of joint CGF $\chi(\lambda)$ is, for $\lambda \in \mathrm{ROA}(\mu)$,

$$H(\lambda) = \left[\frac{\partial^2}{\partial \lambda_m \, \partial \lambda_{\underline{m}}} \chi(\lambda) \right] = \left[\frac{\mu_{m\underline{m}}(\lambda)}{\mu(\lambda)} - \frac{\mu_m(\lambda)}{\mu(\lambda)} \frac{\mu_{\underline{m}}(\lambda)}{\mu(\lambda)} \right] \; . \tag{B-4}$$

There follows, from equations (B-1) and (B-2),

$$\mu_m(\lambda) = \int dz\ z_m \exp(\lambda^T z)\ p(z) \quad \text{for } m=1:M\ ,$$

$$\mu_{m\underline{m}}(\lambda) = \int dz\ z_m\ z_{\underline{m}} \exp(\lambda^T z)\ p(z) \quad \text{for } m,\underline{m}=1:M\ . \qquad \text{(B-5)}$$

All these integrals converge because $\lambda_r \in R_\mu$.

Define (tilted PDF) function

$$\tilde{p}(z,\lambda) = \frac{\exp(\lambda^T z)\ p(z)}{\mu(\lambda)} \quad \text{for } \lambda \in \text{ROA}(\mu)\ , \qquad \text{(B-6)}$$

and define the quantities

$$\tilde{\mu}_m(\lambda) = \int dz\ z_m\ \tilde{p}(z,\lambda) \quad \text{for } m=1:M\ . \qquad \text{(B-7)}$$

Then, the $m,\underline{m}$ element of matrix $H(\lambda)$ is, from equation (B-3),

$$H_{m\underline{m}}(\lambda) = \int dz\ z_m\ z_{\underline{m}}\ \tilde{p}(z,\lambda) - \tilde{\mu}_m(\lambda)\ \tilde{\mu}_{\underline{m}}(\lambda) \quad \text{for } m,\underline{m}=1:M \qquad \text{(B-8)}$$

and $\lambda \in \text{ROA}(\mu)$; that is, $\lambda_r = \text{Re}(\lambda) \in R_\mu$.

Now, let vector λ be real; that is, $\lambda_i = 0$ and $\lambda = \lambda_r \in R_\mu$. It is presumed that there are no linear dependencies among components $\{z_m\}$ of RV z. Also, let real Mx1 vectors

$$\tilde{\mu}(\lambda_r) = [\tilde{\mu}_1(\lambda_r) \cdots \tilde{\mu}_M(\lambda_r)]^T\ , \quad a = [a_1 \cdots a_M]^T\ ,$$

$$v = z - \tilde{\mu}(\lambda_r)\ . \qquad \text{(B-9)}$$

Then, linear combination

$$a^T v \neq 0 \quad \text{for any } a \neq 0 \ , \quad \text{with probability 1} \ . \tag{B-10}$$

Then, the following average must be positive:

$$A(\lambda_r) \equiv E\left\{(a^T v)^2 \ \frac{\exp\left(\lambda_r^T z\right)}{\mu(\lambda_r)}\right\} > 0 \quad \text{for any } a \neq 0 \tag{B-11}$$

because the exp term and $\mu(\lambda_r)$ are always positive. Upon expansion,

$$A(\lambda_r) = a^T E\left[v v^T \exp\left(\lambda_r^T z\right)/\mu(\lambda_r)\right] a \equiv a^T C(\lambda_r) a > 0 \tag{B-12}$$

for any $a \neq 0$. Therefore, MxM matrix $C(\lambda_r)$ is PD for $\lambda_r \in R_\mu$.

The $m,\underline{m}$ element of $C(\lambda_r)$ is, using equations (B-9) and (B-6),

$$C_{m\underline{m}}(\lambda_r) = E\left[v_m \ v_{\underline{m}} \exp\left(\lambda_r^T z\right)/\mu(\lambda_r)\right] =$$

$$= \int dz \ \left[z_m - \tilde{\mu}_m(\lambda_r)\right]\left[z_{\underline{m}} - \tilde{\mu}_{\underline{m}}(\lambda_r)\right] \exp\left(\lambda_r^T z\right) \ p(z)/\mu(\lambda_r) =$$

$$= \int dz \ \left[z_m - \tilde{\mu}_m(\lambda_r)\right]\left[z_{\underline{m}} - \tilde{\mu}_{\underline{m}}(\lambda_r)\right] \ \tilde{p}(z,\lambda_r) =$$

$$= \int dz \ \left[z_m \ z_{\underline{m}} - \tilde{\mu}_m(\lambda_r) \ z_{\underline{m}} - z_m \ \tilde{\mu}_{\underline{m}}(\lambda_r) + \tilde{\mu}_m(\lambda_r) \ \tilde{\mu}_{\underline{m}}(\lambda_r)\right] \tilde{p}(z,\lambda_r) =$$

$$= \int dz \ z_m \ z_{\underline{m}} \ \tilde{p}(z,\lambda_r) - \tilde{\mu}_m(\lambda_r) \ \tilde{\mu}_{\underline{m}}(\lambda_r) \quad \text{for } m,\underline{m}=1:M \ . \tag{B-13}$$

It follows from equation (B-8) that

$$H_{m\underline{m}}(\lambda_r) = C_{m\underline{m}}(\lambda_r) \quad \text{for } m,\underline{m}=1:M \ , \tag{B-14}$$

meaning that $H(\lambda_r) = C(\lambda_r)$ is a PD matrix. That is, HM $H(\lambda)$ of joint CGF $\chi(\lambda)$ is PD for λ real and in R_μ.

APPENDIX C — POSITIVE CURVATURE OF LOGARITHM OF INTEGRAND

Consider the one-dimensional Gaussian RV with MGF and CGF

$$\mu(\lambda) = \exp\left(\frac{1}{2}\sigma^2\lambda^2 + m\lambda\right) ,$$

$$\chi(\lambda) = \ln \mu(\lambda) = \frac{1}{2}\sigma^2\lambda^2 + m\lambda , \qquad (C-1)$$

for all λ; that is, $R_\mu = (-\infty,\infty)$. Then, for evaluation of the PDF at field point z, the logarithm $\Lambda(\lambda,z)$ of the integrand $\Psi(\lambda,z)$ and its derivatives (with respect to λ) are

$$\Lambda(\lambda_r,z) = \chi(\lambda_r) - z\lambda_r = \frac{1}{2}\sigma^2\lambda_r^2 + (m-z)\lambda_r ,$$

$$\Lambda'(\lambda_r,z) = \chi'(\lambda_r) - z = \sigma^2\lambda_r + m - z ,$$

$$\Lambda''(\lambda_r,z) = \chi''(\lambda_r) = \sigma^2 , \quad \text{independent of } z , \qquad (C-2)$$

for all λ_r. Observe that $\Lambda''(\lambda_r,z) > 0$ for all λ_r, provided that $\sigma^2 > 0$; that is, the curvature of $\Lambda(\lambda_r,z)$ is positive for all λ_r. Therefore, $\Lambda(\lambda_r,z)$ can only have a single minimum on the real λ axis, which occurs at $\hat{\lambda}(z) = (z-m)/\sigma^2$.

For the next example, consider the one-dimensional PDF $\exp(-z)$ for $z > 0$. The pertinent λ-domain functions are

$$\mu(\lambda_r) = 1/(1-\lambda_r) , \quad \Lambda(\lambda_r,z) = -\ln(1-\lambda_r) - z\lambda_r ,$$

$$\Lambda'(\lambda_r,z) = 1/(1-\lambda_r) - z , \quad \Lambda''(\lambda_r,z) = 1/(1-\lambda_r)^2 , \qquad (C-3)$$

for $\lambda_r < 1$; that is, $R_\mu = (-\infty,1)$. Again, curvature $\Lambda''(\lambda_r,z) > 0$

for all $\lambda_r < 1$, although the curvature tends to 0+ as $\lambda_r \to -\infty$. Therefore, $\Lambda(\lambda_r,z)$ can, at most, have only one minimum for $\lambda_r < 1$, but it may have none. For example, if $z > 0$, the minimum of $\Lambda(\lambda_r,z)$ is at $\hat{\lambda}(z) = 1 - 1/z$; as $z \to 0+$, then $\hat{\lambda}(z) \to -\infty$.

On the other hand, if $z < 0$, $\Lambda(\lambda_r,z)$ is a strictly monotonically increasing function in R_μ, such that $\Lambda(-\infty,z) = -\infty$, $\Lambda(1-,z) = +\infty$. This last region for z (< 0) corresponds to field points where the exponential PDF is zero; that is, $z < 0$ is an unfeasible point for the PDF, or a point where the PDF is zero.

Both examples above illustrate that if $\Lambda''(\lambda_r,z) > 0$ for all $\lambda_r \in R_\mu$, then $\Lambda(\lambda_r,z)$ can have only one minimum, at most, in that region; that is, in the neighborhood of any point $\lambda_a \in R_\mu$,

$$\Lambda(\lambda_r,z) \cong \Lambda(\lambda_a,z) + \Lambda'(\lambda_a,z)\,(\lambda_r-\lambda_a) + \tfrac{1}{2}\Lambda''(\lambda_a,z)\,(\lambda_r-\lambda_a)^2 \quad . \quad \text{(C-4)}$$

The positiveness of the last coefficient for all $\lambda_a \in R_\mu$ indicates that the function $\Lambda(\lambda_r,z)$ has positive curvature at all points in R_μ.

In M dimensions, with $\lambda = [\lambda_1 \cdots \lambda_M]^T$, equation (C-4) generalizes to

$$\Lambda(\lambda_r,z) \cong \Lambda(\lambda_a,z) + G(\lambda_a,z)^T(\lambda_r-\lambda_a) + \tfrac{1}{2}(\lambda_r-\lambda_a)^T H(\lambda_a,z)(\lambda_r-\lambda_a) \quad \text{(C-5)}$$

for $\lambda_a \in R_\mu$, where Mx1 gradient vector

$$G(\lambda,z) = \left[\frac{\partial}{\partial\lambda_1}\Lambda(\lambda,z) \cdots \frac{\partial}{\partial\lambda_M}\Lambda(\lambda,z)\right]^T \qquad \text{(C-6)}$$

and M×M HM

$$H(\lambda,z) = \left[\frac{\partial^2}{\partial\lambda_m\partial\lambda_{\underline{m}}}\Lambda(\lambda,z)\right], \quad m,\underline{m}=1:M \; . \qquad \text{(C-7)}$$

If matrix $H(\lambda_a,z)$ is PD, the quadratic form in equation (C-5) is positive for all $\lambda_r \neq \lambda_a$; that is, $\Lambda(\lambda_r,z)$ has positive curvature at $\lambda_r = \lambda_a$. If matrix $H(\lambda_r,z)$ is PD for all $\lambda_r \varepsilon R_\mu$, then $\Lambda(\lambda_r,z)$ has positive curvature at all $\lambda_r \varepsilon R_\mu$, and can have, at most, one minimum in this MD region R_μ. Again, this minimum can occur for some components of λ_r equal to infinity, or some components of gradient $G(\lambda_r,z)$ may be nonzero. These latter two cases correspond, respectively, to the situation where the statistical average a of interest is zero, or the integrand $\Psi(\lambda_r,z)$ has no real SP for $\lambda_r \varepsilon R_\mu$.

APPENDIX D — COMPLEX LOCATIONS OF SADDLEPOINTS

Consider PDF $p(z) = \exp(-z)\,[1 + \cos(az)]$ for $z > 0$. This PDF is nonnegative for $z > 0$; although its area is not unity, scaling is irrelevant here. Let $y = 1 - \lambda$ and $b = a^2$. Then the corresponding MGF and CGF and their derivatives are, for $\lambda_r < 1$,

$$\mu(\lambda) = \frac{2y^2+b}{y(y^2+b)}\ , \quad \mu'(\lambda) = \frac{2y^4+by^2+b^2}{y^2(y^2+b)^2}\ , \quad \mu''(\lambda) = \frac{4y^6+6y^2b^2+2b^3}{y^3(y^2+b)^3}\ ,$$

$$\chi'(\lambda) = \frac{\mu'(\lambda)}{\mu(\lambda)} = \frac{2y^4+by^2+b^2}{y(y^2+b)(2y^2+b)}\ ,$$

$$\chi''(\lambda) = \frac{4y^8+7y^4b^2+8y^2b^3+b^4}{y^2(y^2+b)^2(2y^2+b)^2}\ . \qquad \text{(D-1)}$$

Consider the ACs of all these functions, and observe that $\chi''(\lambda_r) > 0$ for all λ_r.

The corresponding properties for the integrand and its logarithm are given by

$$\Psi(\lambda,z) = \mu(\lambda)\exp(-\lambda z)\ ,$$

$$\Lambda(\lambda,z) = \ln \Psi(\lambda,z) = \chi(\lambda) - \lambda z\ ,$$

$$\Lambda'(\lambda,z) = \chi'(\lambda) - z\ , \quad \Lambda''(\lambda) = \chi''(\lambda)\ . \qquad \text{(D-2)}$$

In order to find the SPs of integrand $\Psi(\lambda,z)$, set $\Lambda'(\hat{\lambda},z) = 0$ or $\chi'(\hat{\lambda}) = z > 0$, and obtain the equation

$$2z\hat{y}^5 - 2\hat{y}^4 + 3zb\hat{y}^3 - b\hat{y}^2 + zb^2\hat{y} - b^2 = 0\ , \qquad \text{(D-3)}$$

with $\hat{\lambda} = 1 - \hat{y}$.

As an example, consider a = 1 and z = 0.1. Equation (D-3) then yields the five SPs

$$\hat{\lambda} = -8.8995 \ , \quad 0.4951 \pm i\ 0.70305 \ , \quad 1.4547 \pm i\ 0.68369 \ . \qquad \text{(D-4)}$$

There is one real root and one complex pair of roots within the ROA(μ); that is, with $\lambda_r < 1$. There is another complex pair with $\lambda_r > 1$, which is outside the original ROA(μ). The SP at -8.8995 connects $-i\infty$ to $+i\infty$ directly, by means of a BC. Use of the other four SPs requires using the AC of $\mu(\lambda)$ and the valley at $\lambda = +\infty$, in addition to the residue of the pole of $\Psi(\lambda,z)$ at $\lambda = 1$. Alternatively, the contour could be moved to $\lambda = +\infty$ and the three residues added; the result would be the exact PDF p(z), with no need to resort to SPs or a SPA at all.

An example of a PDF with additional real SPs outside the ROA(μ) is furnished by the sum of three weighted, independent unit-variance exponential RVs; namely, $z = e_1 + e_2/2 + e_3/3$. The PDF and MGF of RV z are

$$p(z) = 3 \exp(-z)\ [1 - \exp(-z)]^2 \quad \text{for } z > 0 \ ,$$

$$\mu(\lambda) = \frac{1}{(1-\lambda)(1-\lambda/2)(1-\lambda/3)} \quad \text{for } \lambda_r < 1 \ . \qquad \text{(D-5)}$$

There follows

$$\Lambda'(\lambda,z) = - z + \frac{1}{1-\lambda} + \frac{1}{2-\lambda} + \frac{1}{3-\lambda} \ . \qquad \text{(D-6)}$$

The numerator of $\Lambda'(\lambda,z)$ is

$$\lambda^3 z + \lambda^2 (3 - 6 z) + \lambda (11 z - 12) + (11 - 6 z) . \qquad (D-7)$$

For example, with $z = 13/12$, the numerator in equation (D-7) has zeros at $\lambda = -1$ and at $\lambda = (55 \pm \sqrt{217})/26 = 1.54881$ and 2.68196, all of which are on the real λ-axis. The latter two SPs are outside the ROA(μ), which is $\lambda_r < 1$. These SPs are the only ones for integrand $\Psi(\lambda,z)$; there are no complex SPs in this example.

If desired, the original BC in the ROA(μ) could be moved so as to pass through the SP of $\Psi(\lambda,z)$ at $\lambda = 1.54881$ when the residue of the AC of $\Psi(\lambda,z)$ at pole location $\lambda = 1$ is also accounted for. Alternatively, the BC could be moved further to the right so as to pass through the SP of $\Psi(\lambda,z)$ at $\lambda = 2.68196$ when the residues at both $\lambda = 1$ and $\lambda = 2$ are also accounted for. Finally, the BC could be moved off to the valley of $\Psi(\lambda,z)$ at $\lambda = +\infty$, giving an exact result for the PDF in terms of the three residues of the AC of $\Psi(\lambda,z)$ at pole locations $\lambda = 1$, 2, and 3.

APPENDIX E — ABSENCE OF SADDLEPOINT EVEN THOUGH z IS A FEASIBLE FIELD POINT

Consider the one-dimensional PDF and MGF pair:

$$p_m(z) = \frac{\exp(-z)}{(1+z)^m} \quad \text{for } z > 0 \ ; \tag{E-1}$$

$$\mu_m(\lambda) = \int_0^\infty dz \, \frac{\exp[-z(1-\lambda)]}{(1+z)^m} = \exp(\omega) \, E_m(\omega) \ , \quad \omega = 1-\lambda \tag{E-2}$$

upon use of reference 3, equation 5.1.4. The integral in equation (E-2) converges for $\lambda_r \leq 1$ for $m > 1$, and for $\lambda_r < 1$ for $m \leq 1$. (Strictly speaking, $p_m(z)$ does not have unit area; its area is $\mu_m(0) = e \, E_m(1)$. However, absolute scaling is not important to the concepts of this appendix.)

A useful recursion is

$$\mu_m(\lambda) = \frac{1}{m-1} \left[1 - \omega \, \mu_{m-1}(\lambda)\right] \quad \text{for } m > 1 \ , \tag{E-3}$$

along with starting values

$$\mu_0(\lambda) = \frac{1}{\omega} \ , \quad \mu_1(\lambda) = \exp(\omega) \, E_1(\omega) \ , \quad \text{both for } \omega_r > 0 \ . \tag{E-4}$$

In particular, for $\omega_r \geq 0$,

$$\mu_2(\lambda) = 1 - \omega \exp(\omega) \, E_1(\omega) \ , \tag{E-5}$$

$$\mu_3(\lambda) = \frac{1}{2}\left[1 - \omega + \omega^2 \exp(\omega) \, E_1(\omega)\right] \ . \tag{E-6}$$

The ACs of all these functions behave as $1/(1-\lambda)$ as $\lambda \to \infty$, according to reference 3, equation 5.1.51.

Additional useful relations are

$$\mu'_m(\lambda) = \mu_{m-1}(\lambda) - \mu_m(\lambda) \ , \qquad \text{(E-7)}$$

$$\mu_m(1) = \frac{1}{m-1} \quad \text{for } m > 1 \ , \qquad \text{(E-8)}$$

$$\mu'_m(1) = \frac{1}{(m-2)(m-1)} \quad \text{for } m > 2 \ . \qquad \text{(E-9)}$$

The derivative of the corresponding CGF is

$$\chi'_m(1) = \frac{\mu'_m(1)}{\mu_m(1)} = \frac{1}{m-2} \quad \text{for } m > 2 \ . \qquad \text{(E-10)}$$

Therefore, $z = 1/(m-2)$ is a breakpoint as to whether a real SP of integrand $\mu_m(\lambda) \exp(-z\lambda)$ can exist for $m > 2$. In addition,

$$\chi''_m(1) = \frac{m-1}{(m-3)(m-2)^2} \quad \text{for } m > 3 \ , \qquad \text{(E-11)}$$

$$\mu''_m(1) = \frac{2}{(m-3)(m-2)(m-1)} \quad \text{for } m > 3 \ . \qquad \text{(E-12)}$$

For $m = 2$, any $z > 0$ is allowed: a real SP always exists. That is, integrand $\Psi(\lambda_r, z) = \mu_2(\lambda_r) \exp(-z\lambda_r)$ always has a minimum for $\lambda_r < 1$, when $z > 0$.

From now on, let $m > 2$. Then, there can be no real SP of

integrand $\Psi(\lambda,z) = \mu_m(\lambda)\exp(-z\lambda)$ if $z > 1/(m-2)$ because of relation (E-10). That is, the curve of $\Psi(\lambda_r,z)$ versus λ_r is still monotonically decreasing at $\lambda_r = 1$ if $z > 1/(m-2)$. This feature is illustrated in figure E-1 for $m = 3$. The slope of $\Psi(\lambda_r,z)$ at $\lambda_r = 1$ is, by use of equations (E-8) and (E-9),

$$\frac{\exp(-z)}{m-1}\left[\frac{1}{m-2} - z\right] \quad \text{for } m > 2 \ . \tag{E-13}$$

Then, the minimum of the integrand (but not a zero slope) is realized at the boundary of definition, namely, at $\lambda_r = 1$. The point $\lambda = 1$ is a BP of the AC of $\mu_m(\lambda)$ because $E_1(\omega)$ has a logarithmic singularity at $\omega = 0$; that is, at $\lambda = 1$. The product $\omega E_1(\omega) = 0$ at $\omega = 0$.

An approach similar to appendix A can be employed here. As an example, consider $m = 3$, as in equation (E-6). The AC of $\mu_3(\lambda)$ has a logarithmic BP at $\lambda = 1$ according to reference 3, equation 5.1.11. Let $z > 0$ and move the BC to a keyhole contour wrapped around $\lambda = 1$ and the positive real axis; this is the steepest descent contour. Then, since $\mu_m(\lambda) \sim 1/(1-\lambda)$ as $\lambda \to \infty$, the integrals on the large circular arcs tend to zero as the radius tends to infinity. Although $E_1(\omega)$ has a logarithmic singularity at $\lambda = 1$, the quantity $(1-\lambda)^2$ tends to zero faster, so that the integral on the small circle around $\lambda = 1$ tends to zero as the radius tends to zero. On the upper (lower) horizontal line, $\lambda_r > 1$, $E_1(1-\lambda) = -\mathrm{Ei}(\lambda-1) +(-)\, i\pi$. Since the integrals are in opposite directions, only the imaginary parts remain, giving the exact result

$$p_3(z) = 2\,\frac{1}{i2\pi}\int_{1+}^{+\infty} d\lambda\, \exp(-z\lambda)\,\frac{(1-\lambda)^2}{2}\,\exp(1-\lambda)\,i\pi = \frac{\exp(-z)}{(1+z)^3}\ . \quad \text{(E-14)}$$

The alternative two-sided PDF $\underline{p}_m(z)$ has a somewhat different behavior:

$$\underline{p}_m(z) = \frac{\exp(-|z|)}{(1+|z|)^m} \quad \text{for all } z\ . \quad \text{(E-15)}$$

The MGF $\underline{\mu}_m(\lambda)$ exists only for $-1 \le \lambda_r \le 1$. In fact,

$$\underline{\mu}_m(\lambda) = \mu_m(\lambda) + \mu_m(-\lambda) \quad \text{for } -1 \le \lambda_r \le 1\ . \quad \text{(E-16)}$$

For m = 3, as λ_r varies in [-1,1], integrand $\underline{\mu}_m(\lambda_r)\exp(-z\lambda_r)$ varies over a <u>finite</u> range; it does not realize a minimum when $|z| > 0.61$. Therefore, no real SP exists for this integrand when $|z| > 0.61$.

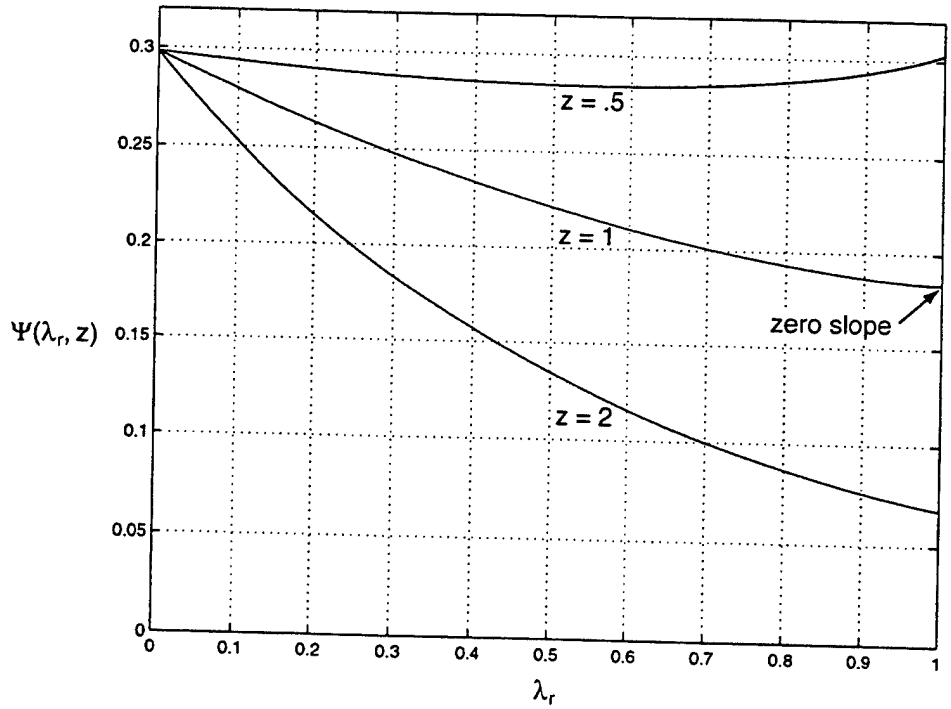


Figure E-1. Integrand of Inverse Laplace Transform for m = 3

APPENDIX F — HESSIAN MATRIX OF ln $\gamma(\lambda)$ IS NONNEGATIVE DEFINITE FOR λ REAL AND IN R_γ

Let g(u) be a real nonnegative scalar function of MD vector u; g(u) need not have finite volume. Define MD $\lambda = \lambda_r + i\,\lambda_i$ and

$$\gamma(\lambda) = \int du\, \exp(-\lambda^T u)\, g(u) \ . \tag{F-1}$$

Let this MD integral converge for $\lambda_r \in R_\gamma$. Define

$$\frac{\partial}{\partial \lambda_m} \gamma(\lambda) = \gamma_m(\lambda) \quad \text{for } m=1:M \ . \tag{F-2}$$

There follows

$$\frac{\partial}{\partial \lambda_m} \ln \gamma(\lambda) = \frac{\gamma_m(\lambda)}{\gamma(\lambda)} \quad \text{for } m=1:M \ , \tag{F-3}$$

and

$$\frac{\partial^2}{\partial \lambda_m\, \partial \lambda_{\underline{m}}} \ln \gamma(\lambda) = \frac{\gamma_{m\underline{m}}(\lambda)}{\gamma(\lambda)} - \frac{\gamma_m(\lambda)}{\gamma(\lambda)} \frac{\gamma_{\underline{m}}(\lambda)}{\gamma(\lambda)} \quad \text{for } m,\underline{m}=1:M \ . \tag{F-4}$$

The MxM HM of ln $\gamma(\lambda)$ is, for $\lambda_r \in R_\gamma$,

$$\tilde{H}(\lambda) \equiv \left[\frac{\partial^2}{\partial \lambda_m\, \partial \lambda_{\underline{m}}} \ln \gamma(\lambda)\right] = \left[\frac{\gamma_{m\underline{m}}(\lambda)}{\gamma(\lambda)} - \frac{\gamma_m(\lambda)}{\gamma(\lambda)} \frac{\gamma_{\underline{m}}(\lambda)}{\gamma(\lambda)}\right] \ . \tag{F-5}$$

From equations (F-1) and (F-2), there follows

$$\gamma_m(\lambda) = -\int du\, u_m \exp(-\lambda^T u)\, g(u) \quad \text{for } m=1:M \tag{F-6}$$

and

$$\gamma_{m\underline{m}}(\lambda) = \int du\, u_m\, u_{\underline{m}} \exp(-\lambda^T u)\, g(u) \quad \text{for } m,\underline{m}=1:M \ . \tag{F-7}$$

Now, for $\mathrm{Re}(\lambda) \in R_\gamma$, define (tilted) function

$$\tilde{g}(u,\lambda) = \frac{\exp(-\lambda^T u)\, g(u)}{\gamma(\lambda)} \qquad \text{(F-8)}$$

and the quantities

$$\tilde{v}_m(\lambda) = \int du\, u_m\, \tilde{g}(u,\lambda) \quad \text{for } m=1:M\ . \qquad \text{(F-9)}$$

Then, the $m,\underline{m}$ element of HM $\tilde{H}(\lambda)$ in equation (F-5) becomes

$$\tilde{H}_{m\underline{m}}(\lambda) = \int du\, u_m\, u_{\underline{m}}\, \tilde{g}(u,\lambda) - \tilde{v}_m(\lambda)\, \tilde{v}_{\underline{m}}(\lambda) \quad \text{for } m,\underline{m}=1:M\ . \qquad \text{(F-10)}$$

Observe that the function $\tilde{g}(u,\lambda)$ defined in equation (F-8) has unit volume in u for $\lambda = \lambda_r \in R_\gamma$; that is,

$$\int du\, \tilde{g}(u,\lambda_r) = 1 \quad \text{for all } \lambda_r \in R_\gamma\ . \qquad \text{(F-11)}$$

Thus, $\tilde{g}(u,\lambda_r)$ has the properties of a legal PDF of argument u for any $\lambda_r \in R_\gamma$; in particular, $\tilde{g}(u,\lambda_r)$ is real, nonnegative, and has unit volume in MD u space. Therefore, considering $\tilde{g}(u,\lambda_r)$ as a MD PDF in u, the mean of the corresponding m-th RV $u_m(\lambda_r)$ is

$$\tilde{v}_m(\lambda_r) = \int du\, u_m\, \tilde{g}(u,\lambda_r) \quad \text{for } m=1:M\ , \qquad \text{(F-12)}$$

while the covariance of the $m,\underline{m}$ pair of RVs $u_m(\lambda_r)$, $u_{\underline{m}}(\lambda_r)$ is

$$\tilde{U}_{m\underline{m}}(\lambda_r) = \int du\, u_m\, u_{\underline{m}}\, \tilde{g}(u,\lambda_r) - \tilde{v}_m(\lambda_r)\, \tilde{v}_{\underline{m}}(\lambda_r) \quad \text{for } m,\underline{m}=1:M. \qquad \text{(F-13)}$$

But, the covariance matrix of any PDF is always nonnegative definite because with random vector $w = u(\lambda_r) - v(\lambda_r) =$

$[u_1(\lambda_r) - \tilde{v}_1(\lambda_r) \cdots u_M(\lambda_r) - \tilde{v}_M(\lambda_r)]^T$, the average

$$E\ (a^T w)^2 = a^T\ E(w\ w^T)\ a = a^T\ \tilde{U}(\lambda_r)\ a \geq 0 \qquad \text{(F-14)}$$

for any Mx1 real vector a. $\tilde{U}(\lambda_r) = [\tilde{U}_{m\underline{m}}(\lambda_r)]$ is the MxM covariance matrix of RV $u(\lambda_r)$ and must be nonnegative definite.

Comparison of equations (F-10) and (F-13) reveals that

$$\tilde{H}_{m\underline{m}}(\lambda_r) = \tilde{U}_{m\underline{m}}(\lambda_r) \quad \text{for } m,\underline{m}=1:M\ . \qquad \text{(F-15)}$$

Therefore, matrix $\tilde{H}(\lambda_r) = \tilde{U}(\lambda_r)$ is nonnegative definite; that is, the HM $\tilde{H}(\lambda)$ of ln $\gamma(\lambda)$ is nonnegative definite for λ real and in R_γ.

As a first example, consider

$$g(u) = \delta(u_1 - z_1) \cdots \delta(u_M - z_M)\ , \qquad \text{(F-16)}$$

for which

$$\gamma(\lambda) = \exp(-\lambda^T\ \lambda) \quad \text{for all } \lambda\ . \qquad \text{(F-17)}$$

The HM of ln $\gamma(\lambda)$ $(= -\lambda^T\ z)$ is zero for all λ; this HM has M zero eigenvalues and is nonnegative definite for all λ.

The second example is

$$g(u) = \delta(u_1 - z_1)\ U(u_2 - z_2) \cdots U(u_M - z_M)\ , \qquad \text{(F-18)}$$

for which

$$\gamma(\lambda) = \exp(-\lambda^T z) \Big/ \prod_{m=2}^{M} (\lambda_m) \quad \text{for } \mathrm{Re}(\lambda_m) > 0 \ , \ m=2:M \ , \qquad \text{(F-19)}$$

and λ_1 is arbitrary. There follows

$$\ln \gamma(\lambda) = -\lambda^T z - \sum_{m=2}^{M} \ln(\lambda_m) \ , \qquad \text{(F-20)}$$

for which the gradient vector is

$$G(\lambda) = [-z_1 \ -z_2 - 1/\lambda_2 \ \cdots \ -z_M - 1/\lambda_M]^T \ . \qquad \text{(F-21)}$$

The MxM HM of $\ln \gamma(\lambda)$ is then

$$\tilde{H}(\lambda) = \mathrm{diag}\left[0 \ 1/\lambda_2^2 \ \cdots \ 1/\lambda_M^2\right] \ . \qquad \text{(F-22)}$$

The matrix $\tilde{H}(\lambda_r)$ has one zero eigenvalue and M-1 positive eigenvalues; thus, $\tilde{H}(\lambda_r)$ is nonnegative definite.

REFERENCES

1. A. H. Nuttall, "Saddlepoint Approximation and First-Order Correction Term to the Joint Probability Density Function of M Quadratic and Linear Forms in K Gaussian Random Variables with Arbitrary Means and Covariances," NUWC-NPT Technical Report 11,262, Naval Undersea Warfare Center Division, Newport, RI, 12 December 2000.

2. A. H. Nuttall, "Saddlepoint Approximations for Various Statistics of Dependent Non-Gaussian Random Variables: Applications to the Maximum Variate and the Range Variate," NUWC-NPT Technical Report 11,280, Naval Undersea Warfare Center Division, Newport, RI, 10 April 2001.

3. **Handbook of Mathematical Functions**, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series, Number 55, U.S. Government Printing Office, Washington, DC, June 1964.

4. Peter M. McCullagh, **Tensor Methods in Statistics**, Chapman and Hall, London, England, 1987.

INITIAL DISTRIBUTION LIST

Addressee	No. of Copies
Center for Naval Analyses, VA	1
Coast Guard Academy, CT	
J. Wolcin	1
Commander Submarine Force, U.S. Pacific Fleet, HI	
W. Mosa, CSP N72	1
Defense Technical Information Center	2
Griffiss Air Force Base, NY	
Documents Library	1
J. Michels	1
Hanscom Air Force Base, MA	
M. Rangaswamy	1
National Radio Astronomy Observatory, VA	
F. Schwab	1
National Technical Information Service, VA	10
Naval Environmental Prediction Research Facility, CA	1
Naval Intelligence Command, DC	1
Naval Oceanographic and Atmospheric Research Laboratory, MS	
E. Franchi	1
R. Wagstaff	1
Naval Oceanographic Office, MS	1
Naval Postgraduate School, CA	
Superintendent	1
C. Therrien	1
Naval Research Laboratory, DC	
W. Gabriel	1
E. Wald	1
Naval Surface Warfare Center, FL	
E. Linsenmeyer	1
D. Skinner	1
Naval Surface Warfare Center, VA	
J. Gray	1
Naval Technical Intelligence Center, DC	
Commanding Officer	1
D. Rothenberger	1
Naval Undersea Warfare Center, FL	
Officer in Charge	1
Naval Weapons Center, CA	1
Office of the Chief of Naval Research, VA	
ONR 321 (D. Johnson)	1
ONR 321US (J. Tague)	1
ONR 322 (R. Tipper)	1
ONR 334 (P. Abraham)	1
ONR 31 (R. R. Junker)	1
ONR 311 (A. M. van Tilborg)	1
ONR 312 (M. N. Yoder)	1
ONR 313 (N. L. Gerr)	1
ONR 32 (S. E. Ramberg	1
ONR 321 (F. Herr)	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
Office of Naval Research	
ONR 33 (S. G. Lekoudis)	1
ONR 334 (A. J. Tucker)	1
ONR 342 (W. S. Vaughan)	1
ONR 343 (R. Cole)	1
ONR 362 (M. Sponder)	1
U.S. Air Force, Maxwell Air Force Base, AL	
Air University Library	1
Vandenberg Air Force Base, CA	
CAPT R. Leonard	1
Brown University, RI	
Documents Library	1
Catholic University of America, DC	
J. McCoy	1
Drexel University, PA	
S. Kesler	1
Duke University, NC	
J. Krolik	1
Harvard University, MA	
Gordon McKay Library	1
Johns Hopkins University, Applied Physics Laboratory, MD	
H. M. South	1
T. N. Stewart	1
Lawrence Livermore National Laboratory, CA	
L. Ng	1
Los Alamos National Laboratory, NM	1
Marine Biological Laboratory, MA	
Library	1
Massachusetts Institute of Technology, MA	
Barker Engineering Library	1
Massachusetts Institute of Technology, Lincoln Laboratory, MA	
V. Premus	1
J. Ward	1
Pennsylvania State University, Applied Research Laboratory, PA	
R. Hettche	1
E. Liszka	1
F. Symons	1
D. Abraham	1
Princeton University, NJ	
S. Schwartz	1
Rutgers University, NJ	
S. Orfanidis	1
San Diego State University, CA	
F. Harris	1
Scripps Institution of Oceanography, Marine Physical Laboratory, CA	
Director	1
Syracuse University, NY	
D. Weiner	1
Engineering Societies Information Center, Kansas City, MO	
Linda Hall Library-East	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
University of Colorado, CO	
L. Scharf	1
University of Connecticut, CT	
Wilbur Cross Library	1
C. Knapp	1
P. Willett	1
University of Florida, FL	
D. Childers	1
University of Hartford	
Science and Engineering Library	1
University of Illinois, IL	
D. Jones	1
University of Illinois at Chicago, IL	
A. Nehorai	1
University of Massachusetts, MA	
C. Chen	1
University of Massachusetts Dartmouth, MA	
J. Buck	1
University of Michigan, MI	
Communications and Signal Processing Laboratory	1
W. Williams	1
University of Minnesota, MN	
M. Kaveh	1
University of Rhode Island, RI	
Library	1
G. Boudreaux-Bartels	1
S. Kay	1
D. Tufts	1
University of Rochester, NY	
E. Titlebaum	1
University of Southern California, CA	
W. Lindsey	1
A. Polydoros	1
University of Texas, TX	
Applied Research Laboratory	1
C. Penrod	1
University of Toronto	
N. Reid	1
University of Washington, WA	
Applied Physics Laboratory	1
D. Lytle	1
J. Ritcey	1
R. Spindel	1
Villanova University, PA	
M. Amin	1
Woods Hole Oceanographic Institution, MA	
Director	1
T. Stanton	1
Yale University, CT	
Kline Science Library	1
P. Schultheiss	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
Analysis and Technology, CT	
Library	1
Analysis and Technology, VA	
D. Clark	1
Atlantic Aerospace Electronics Corp.	
R. Stahl	1
Bell Communications Research, NJ	
D. Sunday	1
Berkeley Research, CA	
S. McDonald	1
Biosgroup	
R. Perline	1
BBN Technologies, CT	
P. Cable	1
Bolt, Beranek, and Newman, MA	
H. Gish	1
DSR, Inc., VA	
M. Bozek-Kuzmicki	1
EG&G Services, CT	
J. Pratt	1
Engineering Technology Center	
D. Lerro	1
General Electric, NJ	
H. Urkowitz	1
Harris Scientific Services, NY	
B. Harris	1
Lincom Corporation, MA	
T. Schonhoff	1
Lockheed Martin, Undersea Systems, VA	
M. Flicker	1
Lockheed Martin, Ocean Sensor Systems, NY	
R. Schumacher	1
Marconi Aerospace Defense Systems, TX	
R. D. Wallace	1
MITRE Corporation, VA	
S. Pawlukiewicz	1
R. Bethel	1
Neural Technology, Inc., SC	
E. A. Tagliarini	1
Orincon Corporation, VA	
H. Cox	1
Prometheus, RI	
M. Barrett	1
J. Byrnes	1
Raytheon, RI	
R. Conner	1
S. Reese	1
Schlumberger-Doll Research, CT	
R. Shenoy	1
Science Applications International Corporation, CA	
C. Katz	1

INITIAL DISTRIBUTION LIST (Cont'd)

Addressee	No. of Copies
Science Applications International Corporation, VA	
P. Mikhalevsky	1
Toyon Research, CA	
M. Van Blaricum	1
Tracor, TX	
T. Leih	1
Undersea Sensor Systems, Inc., IN	
R. Kenefic	1
Westinghouse Electric, Annapolis, MD	
H. Newman	1
Westinghouse Electric, Baltimore, MD	
R. Park	1
K. Harvel, Austin, TX	1